using two sets of cards, each of the first set punched with a value of a^3+b^3 and of the second set with the difference between two successive values of $2c^3+2d^3$. The same field is used for the two variables. Figure 4 shows part of one record made by placing a single card of the first set at the beginning of the second set, and accumulating in column III the totals after the successive cards. For convenience in verifying, columns I and II are included in the record, the former being merely a serial number punched in the cards. The latter column is headed by $(22)^3+(9)^3$, and contains thereafter the successive differences referred to above. The numbers in column III, therefore, are the sums,

$$(22)^3 + (9)^3 + 2c^3 + 2d^3,$$
$$c = 0,1,2, \cdots; d = 0,1,2, \cdots; c \geqq d.$$

The sorting machine separates punched cards into groups ordered according to the position of the holes in any column. The sorting is on only one column at a time. If the field covers two or more columns, successive sortings are made on them from right to left. If one column constitutes the entire field of a variable, a single sorting yields the frequency distribution of that variable. Figure 5 shows cards sorted into two such distributions. In the lower one, the independent variable is the "number of pigs in a litter," the sample comprising 2789 litters of Poland China pigs. This set of cards was selected for the illustration because two extra positions were punched in a single-column field, making twelve classes instead of the usual ten. The cards in the thirteenth class on the extreme right were unpunched. A column utilized in this manner can be used only for sorting because, although the sorter is

I	II	III
541	11377	11377
1	2	11379
2	2	11381
3	12	11393
4	2	11395
5	14	11409
6	22	11431
7	2	11433
8	14	11447
9	38	11485
10	20	11505
11	2	11507
12	14	11521
13	38	11559
14	68	11627
15	2	11629
16	4	11633
17	10	11643
18	38	11681
19	74	11755
20	54	11809
21	2	11811
22	14	11825
23	38	11863
24	14	11877
25	60	11937
26	122	12059
27	4	12063
28	2	12065
29	14	12079
30	38	12117
31	74	12191
32	50	12241
33	72	12313
34	88	12401
35	2	12403
36	14	12417
37	38	12455
38	40	12495
39	34	12529
40	122	12651
41	98	12749
42	84	12833

FIGURE 4

capable of handling this maximum of thirteen groups in any one column, the tabulating machine adds only such numbers as are punched in the ten positions, 0, 1, 2, · · · , 9.

After the illustrated sorting the cards are transferred to the tabulator for the determination of the class frequencies. This is accomplished by a "card count" connection causing the addition of one unit for each card passing, independent of the position of the hole punched in any field. The result is the following frequency distribution:-

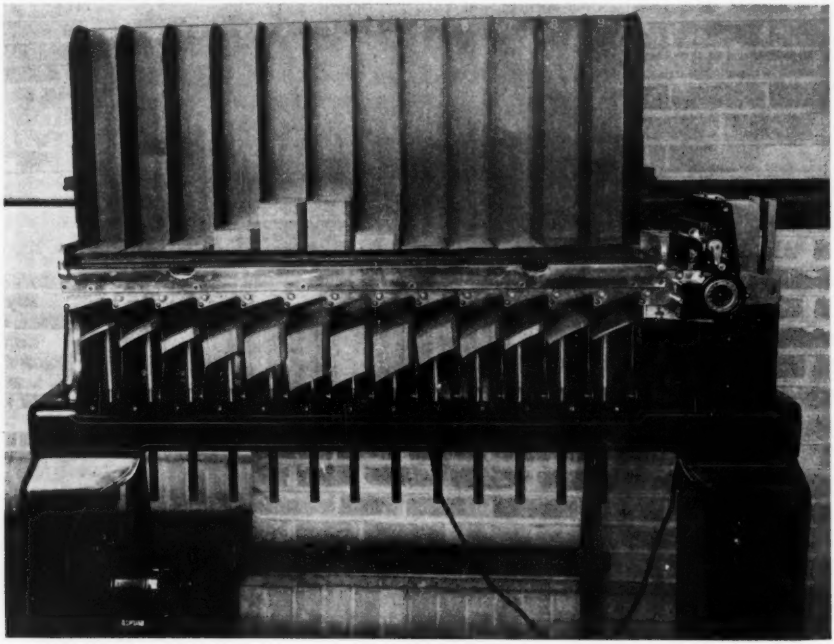


FIGURE 5

Number pigs in litter	2–3	4	5	6	7	8	9	10	11	12	13	14	15–17
Frequency	29	59	100	232	392	530	469	419	263	164	76	35	21

The constants of the distribution are now computed in the usual manner. The distribution illustrated in the upper part of Fig. 5 is that of 1352 horses sorted on the coded values of the variable, "heart girth," appearing in column 18 of the cards illustrated in Figs. 2 and 3. The records were furnished by the

Horse Association of America as collected from the horse-pulling contests conducted during 1926, and have been studied by Mr. A. E. Brandt[1] of our department.

Most of the statistical sets coming into our laboratory contain two or more variables, usually continuous. In the preliminary treatment such of the variables as may require it are coded before they are punched in the cards. This process differs only in minor details from that usually employed in grouping data into frequency distributions and preparing the tables for computation, but some of the likenesses and differences are important enough to warrant explanation. The first step in the two processes is identical,—for each variable, A', B', C', $\cdots$, a class interval is selected, and such interval then serves as a unit of measurement. The second step in the usual procedure is the tallying of each observed value in the corresponding interval, or of each pair of correlated values in the appropriate cell; and this is followed by the third step, the choice of an origin for each variable, usually at a class mark near the mode. (The variables now having each a new origin and a new unit of measurement may be represented by A, B, C, $\cdots$). In the punched card system, the latter of these steps is the second, the origin being customarily fixed at the lowest class mark of the sample, and the coded values, A, B, C, $\cdots$ being punched in the cards. Finally, the tallying process, designated as step two above, is replaced by the sorting of the cards on the several variables in succession.

The total-printing and listing tabulating machine, receiving the cards after they are sorted on one of the variables, say A, records the following information as illustrated in block A of Fig. 6; first, the successive coded values of A, column I; second, the frequency, f_A, of each A-class, column II; third, the sum of the values, corresponding to the several A-classes, of each of the variables, i.e., $(\sum A)_A$, $(\sum B)_A$, etc., columns III, IV and V. This is all done simultaneously during a single run of the cards through the machine. After successive sorts are made on the other variables in turn, similar records are made as shown in blocks B and C.[2] The identity of the totals in any column constitutes one of the checks on the accuracy of the machine work.

In Fig. 6 are found all the data required for the calculation of the means, standard deviations, correlation coefficients and correlation ratios. The formulas[3] are as follows:

---

[1] *A preliminary study of the relation between form and power in the horse*, a paper read before the Am. Soc. Ag. Eng., at Saint Paul, 1927.

[2] These data are from A. E. Brandt, *Effect of breed and body measurements on dressing percent in swine*, Master's thesis, Iowa State College, 1926. Variable A is "length," B is "heart girth," and C is "live weight."

[3] Formulas adapted from J. A. Harris, American Naturalist, vol. 44 (1910), pp. 693-699; and from J. Blakeman, Biometrika, vol. 4 (1906), pp. 332–350.

	I	II	III	IV	V
	A	f_A	$(\sum A)_A$	$(\sum B)_A$	$(\sum C)_A$
		6		3	6
	1	30	30	47	62
	2	57	114	119	146
	3	69	207	238	225
	4	78	312	278	269
A	5	33	165	147	137
	6	12	72	67	70
	7	4	28	22	23
	9	2	18	16	18
		291	946	937	956
	B	f_B	$(\sum A)_B$	$(\sum B)_B$	$(\sum C)_B$
		16	18		17
	1	34	76	34	71
	2	69	188	138	172
	3	59	190	177	177
	4	33	129	132	122
B	5	44	173	220	190
	6	22	101	132	113
	7	11	51	77	72
	9	3	20	27	22
		291	946	937	956
	C	f_C	$(\sum A)_C$	$(\sum B)_C$	$(\sum C)_C$
		1	2		
	1	21	22	11	21
	2	76	195	147	152
	3	87	283	250	261
	4	53	197	231	212
C	5	29	127	150	145
	6	12	52	67	72
	7	6	31	38	42
	8	3	13	20	24
	9	3	24	23	27
		291	946	937	956

FIGURE 6

$$N = \sum f_A, \quad M_A = (\sum A)/N, \quad \sigma_A^2 = [\sum A^2 - (\sum A)M_A]/N,$$

$$r_{AB} = [\sum AB - (\sum A)M_B]/[\sum A^2 - (\sum A)M_A]^{1/2}[\sum B^2 - (\sum B)M_B]^{1/2},$$

$$\eta_{BA}^2 = \left[\sum \frac{(\sum B)_A^2}{f_A} - (\sum B)M_B\right]/[\sum B^2 - (\sum B)M_B].$$

A few particulars will serve to make the whole process obvious. First, each of the totals in column III is $\sum A$, each of those in column IV is $\sum B$, and so on. Secondly, in columns I and IV, block A, the pairs of numbers in each line, upon multiplication and addition, yield the product-sum, $\sum AB$, because $\sum AB = \sum A(\sum B)_A$. The correctness of the calculation is verified by using columns I and III, block B. Thirdly, by the same multiplication-addition process, $\sum A^2$ is obtained from columns I and III, block A, and verified by calculating $\sum f_A A^2$ directly from columns I and II. Fourthly, $\sum(\sum B)_A^2/f_A$, required for the computation of η_{BA} is taken from columns II and IV block A, and verified by computing this sum as $\sum(\sum B)_A[(\sum B)_A/f_A]$. For economy of effort and time the computations should be completed in some such systematic form as that indicated by B. B. Smith[1] or by Wallace and Snedecor.[2]

If desired, the computations may be verified by using another variable, $S = A + B + C + \cdots$, whose values are computed from the coded values of the observed variables and punched in each card. A sixth column would be added to the form illustrated in Fig. 6, but complete verification would be obtained with the following parts of the record deleted:—column III, block A; columns III and IV, block B; and columns III, IV, V, block C. The corresponding correlation ratios are sacrificed.

The multiple correlation constants are best obtained by direct solution of the normal equations, using either the method of Gauss[3] or one of the adaptations.[4] An idea of the rapidity with which such constants may be computed is conveyed in this statement; one of our operators, starting with the simple correlation coefficients among eight variables, including the sum, completes in forty minutes the regression equation and the multiple correlation coefficient.[5]

---

[1] *The use of punched card tabulating equipment in multiple correlation problems*, a mimeographed pamphlet issued by Bureau of Ag. Ec., U.S.D.A., 1923.

[2] *Correlation and machine calculation*, Iowa State College Bulletin, vol. 23, No. 35, 1925.

[3] *Disquisitio de Elementis Ellipticio Palladis*, Societati Regiae Tradita, vol. 25, 1810. See Carl Friedrich Gauss, Werke, vol. 6, p. 21.

[4] M. H. Doolittle, U.S. Coast and Geodetic Survey Report (1878), p. 115.
Tolley and Ezekiel, Journal of the American Statistical Association, vol. 18 (1923), p. 993.
Wallace and Snedecor, loc. cit.

[5] Since this was written there has come to hand an article advocating the direct solution of the normal equations. See Tolley and Ezekiel, *The Doolittle vs. the Kelley-Salisbury method*, Journal of the American Statistical Association, vol 22 (1927), p. 497.

The punched card equipment is well adapted to the study of multiple curvilinear regression using the method of Ezekiel.[1] The errors of estimate, e, are punched in the corresponding cards. The cards are then sorted successively on the several independent variables. After the sort on A, for example, the errors of estimate are tabulated for each coded value of A, yielding $(\sum e)_A$. The means, $(\sum e)_A/f_A$, are then plotted as deviations from the line,

$$\overline{X} = M_X + b_{XA \cdot BC} \ldots (A - M_A),$$

where X is the dependent variable, and $\overline{X}$ its value estimated from the complete linear regression equation. This is done for each independent variable. A curve is then fitted to each of the sets of points so plotted, and these curves are made the basis of the first approximations to the assumed curvelinear values of X. The new errors of estimate, e', are computed, and the stage is now set for the second approximation.

To convey an idea of the uses to which the punched card equipment has been put during the first year of its installation at Iowa State College, there is appended a partial list of the problems studied, not including those already cited in this article;

1. Relation of college grades to high school averages, mental tests, physical condition and other characters.
2. Correlation among mental test grades.
3. Butter fat production as influenced by temperature, advancement of the gestation and lactation periods, etc.
4. Cost per hundred pounds gain in swine as related to season, feeding system, average daily gain, and initial weight.
5. Differences in cattle prices at the Omaha, Kansas City, and Chicago markets.
6. Analysis of demand meter records obtained in farm electric installations.
7. Results of the Iowa corn contests.
8. Inheritance of characters in corn.
9. Inheritance in tomatoes.
10. Relations among number of stalks per hill, number of ears per stalk, number of ears per hill, size of ears, fodder weight, and yield in sweet corn.
11. Correlations among number of plants and fruits per hill and size of fruits in strawberries.
12. Results of high school contest in judging grains and animals.
13. Biological significance of partial regression coefficients from the standpoint of probable errors.

---

[1] Journal of the American Statistical Association, vol. 19 (1924), p. 431.

14. Magnitude of arithmetical changes in statistical constants due to grouping.
15. Relations among factors affecting farm profits.
16. Man labor and horse labor used in various farm enterprises and operations.
17. Farm mortgage study.
18. Livestock shipping associations survey.
19. Bank and credit conditions in Iowa.
20. Land and crop utilization as related to type of farming.

THE LAW OF SMALL NUMBERS[1]

By A. R. CRATHORNE, University of Illinois

In his *Sur la probabilité des jugements*, published in 1837, Poisson discussed in his chapters on the law of large numbers the probability that the number of happenings of an event E in n trials will fall within certain assigned limits. Letting p be the probability of success, he starts with the binomial expansion

$$(p+q)^n = p^n + np^{n-1}q + \frac{n\cdot(n-1)}{2!}p^{n-2}q^2 + \cdots,$$

the terms of which give the probability of the success of E, n times, $(n-1)$ times, and so on, out of n trials. In general the probability P_x of success x times out of n trials is given by the $(n-x+1)$th term of the series

$$\frac{n(n-1)\cdots(n-x+1)}{x!}p^xq^{n-x}.$$

To avoid the large factorials in this expression he reduced the problem to the consideration of the integral $\int e^{-t^2}dt$. In doing this it was necessary to assume not only that n was large but also to assume that p was not too small. The general idea underlying this treatment of the problem was of course not original with Poisson. What Poisson offered that was new at this point was a special consideration of the case where p was small, n still remaining large. He showed that if n was allowed to increase indefinitely and p to decrease towards zero in such a way that np remained constant, the expression

$$P_x = \frac{n(n-1)\cdots(n-x+1)}{x!}p^xq^{n-x} \quad \text{approaches} \quad P_x = \frac{m^xe^{-m}}{x!},$$

[1] Read at the twelfth annual meeting of the Association, Nashville, December 30, 1927.

where $m=np$ and x is not a large number. This gives the probability of x successes in n trials when p is too small for the more general formula.

This result was more or less a by-product of Poisson's discussion of the law of large numbers and seems to have received little attention until 1898 when Bortkewitsch published his "Gesetz der kleinen Zahlen," in which he called attention to Poisson's formula and emphasized the very decided difference between the behavior of small and of large frequencies. He pointed out the facts that small frequencies from large fields tend to fit the norm given by Poisson's formula and that they have a certain stability of their own. Bortkewitsch called this realization in statistical experience the law of small numbers.

If the frequencies for various values of x in a series of observations are given by the expression $(m^x e^{-m})/x!$, we have a maximum for $x=m$ where m is the mean. There is a rather rapid decrease on either side as x takes on integral values. Hence the observations never vary very much from their mean value and may be considered as quite stable. It should be emphasized that the above expression is defined only for integral values of x.

The discussion by Bortkewitsch is somewhat as follows: If we are investigating a series of observations of some phenomena, it is important to know whether the variations in the observations can be explained by the theory of probability and likened to some game of chance. Consider an event E with a probability of happening p. Out of n trials, E happens s_1 times, in another series of n trials, E happens s_2 times, and so on. In this way we get a series of numbers $s_1, s_2, s_3 \cdots$. In a large number of s's we shall find that the number $s=np$ occurs more than any other. If we mark off an interval on each side of np so that the double interval contains one half the s's, then it can be shown that this interval is $0.6745\ [np(1-p)]^{1/2}$, the so called probable deviation. If now we consider a series of relative numbers coming from actual observation and mark off on each side of the mean an interval in which half of the observations lie, and if we find that this interval is the same as the one given by some urn problem, then we are justified in investigating further as to whether a constant probability underlies the observed phenomena. In other words a constant complex of causes probably underlies the phenomena. It is, of course, not necessary to use the interval about the mean containing one half of the observations. Some other fractional part would serve. For most purposes the so-called standard deviation is preferable.

In his special case, Bortkewitsch considers the numbers $x_1, x_2, \cdots, x_n$ which come from observation and represent the number of times a rare event has happened, say during a series of years. For example, it may be the number of triplets born in a country each year for as many years as we have information; or it may be the number of alpha particles colliding with a small screen as in the Rutherford and Geiger experiment; or again it may be the number of

triple plays per year in one of our national leagues. If the frequency distribution of the x's fit Poisson's formula, then it is easy to show that the standard deviation of the x's is $\sigma_x = (np)^{1/2} = m^{1/2}$. Bortkewitsch calls this the calculation of the standard deviation by the indirect method. Since the theoretical m is not known, an approximation m', the mean of the x's, is used. Then computing the standard deviation directly from the observations, we have

$$\sigma_x' = \left[\sum \frac{(x_i - m')^2}{n}\right]^{1/2}.$$

If σ_x and σ_x' are statistically equal, then according to Bortkewitsch we are justified in saying that the theoretical formula fits the actual observations. Of course σ_x and σ_x' are accompanied by errors and do not necessarily have to be exactly equal. Bortkewitsch showed that the standard deviation of σ_x^2 is is $(m'/n)^{1/2}$ and of $\sigma_x'^2$ is $[(2m'^2+m')/n]^{1/2}$.

To give one example, the number of quadruplets born in Prussia each year during a period of 69 years of observation is given by Schmidt. In 14 of the years no quadruplets were born; in 24 years, 1 each year; in 17 years, 2 each year; and so on according to the following table:

x	Observed	Theoretical
0	14	14.2
1	24	22.5
2	17	17.7
3	9	9.3
4	2	3.7
5	2	1.2
6	1	0.3
7	0	0.1
	69	69

In the table, the first column gives, the number, x, of quadruplets, the second column gives the number of years in which x quadruplets were actually born, the third column gives the theoretical results from Poisson's formula where $m = 1.58$.

From these figures we find $\sigma_x^2 = 1.58$, $\sigma_x'^2 = 1.72$. The standard deviation of $\sigma_x'^2$ is 0.31. The difference $1.72 - 1.58 = .14 < .31$ is then not so much as might be expected.

In most statistical problems of the above type dealing with events that are more common, the difference between the σ_x and the σ_x' is more marked. The σ_x' may be quite a large multiple of σ_x. Bortkewitsch states this fact by saying that rare events are more stable than other events.

This apparent stability of series of small numbers has an explanation in the so-called Lexian theory. Lexis pointed out that the variations of certain series of observations which he called typical may have at their foundation a variable probability. Consider a large pile of thoroughly mixed black and white balls. From the pile m urns are filled say by dipping the urns into the pile. Let p_0 represent the proportion of white balls in the pile. Due to accidental variations

the proportion of white balls in the urns will vary from urn to urn. We may say then that the probabilities of drawing a white ball from the urns are $p_1, p_2, \cdots, p_m$ and that the mean of the p's is an approximation to p_0. Take n drawings from each urn, replacing the ball after each drawing.

Let s represent the proportion of white balls drawn. We shall have a series of numbers $s_1, s_2, \cdots, s_n$. According to the theory of probabilities, the standard deviation of the s's is $R=(r^2+a^2)^{1/2}$, where $r=(p_0q_0/n)^{1/2}$ is the standard deviation if each urn contains the same proportion, p_0, of white balls. This Lexis called the unessential component of the fluctuation. The a is the standard deviation of the variable probabilities p and is equal to $[(1/m)\sum(p_i-p_0)^2]^{1/2}$ and is called the physical component. Lexis then used for his criterium of dispersion the quotient $Q=R/r$. When $Q=1$, he called the dispersion normal; and when $Q>1$, supernormal. For $Q=1$ we have a maximum of stability and for any series of observations for which this condition is filled we may say that there is a constant complex of causes which may be analogous to some game of chance.

Substituting the above values of r and a we find

$$Q=\left[1+\frac{n\sum(p_k-p_0)^2}{mp_0q_0}\right]^{\frac{1}{2}}.$$

Q may be made to approach unity by decreasing n, the number of drawings. In statistical phraseology this is the same as restricting the field of observations more and more. Q may also approach 1 if (p_i-p_0) is small. For $p_i=p_0$ we have normal dispersion. It is usually not possible to know from just the observations whether (p_i-p_0) is small or not; but if p_i and p_0 are both small, their difference is small. Other things being equal we may then say that the smaller the p's the less the value of Q and, accordingly, the less the dispersion and the greater the stability of the series. To the two methods of increasing stability we may add combinations of the two.

In a statistical problem we do not have the convenient urn but have a vaguer term, field of observation. For example, in statistics concerned with male births from year to year in a district, n will be the total number of births and m the number of years under observation. n is easy to find here, but it is not usually so easy, and an essential difficulty arises. For example, our statistics may be a list of the number of cases of death from Addison's disease in the country for a series of years. Shall we consider our field of observation to include the whole population or the whole population of adults? Or shall we limit it to the total number of deaths during the year, or perhaps to those dying of disease? Or we may be considering the number of bachelors who marry when over seventy. Shall we consider our field to be all bachelors, or all bachelors over seventy, or merely all exposed bachelors? What is n in the case of

alpha particles striking a small screen? It is due to Bortkewitsch that we can give a theoretical representation of the frequency of such rare events without knowing anything about the number in the field of observation except that it is a large number. All we know is an empirical estimate of $m' = np$.

The theory of statistics is lucky in having among its workers vigorous and sometimes caustic writers who are not altogether impersonal. This has livened up and invigorated some otherwise rather formal discussions. Bortkewitsch, with his law of small numbers, has done his share in furnishing material for these writers, and his answers have been equally pungent.

The name itself, "Bortkewitsch's law of small numbers," has been thoroughly criticised. It has been said that there is no law; that if there were a law, it was not Bortkewitsch's; that it was not about numbers; and if it were the numbers were not small. In mathematical statistics we have a well known and fundamental group of propositions which make up what is called the law of large numbers. This law is the bed rock upon which the greater part of the theory is built. We are accustomed to think of small numbers as interlopers and we ruthlessly cast them out. So it is somewhat startling to find that they claim to be law abiding even if it is to a code of laws of their own making. But the law of small numbers is not something opposed to the law of large numbers or something to be set up alongside and compared with it, but it is part and parcel of the older fundamental law and was so considered by Poisson. A great deal of confusion seems to have centered about the term "small numbers," as if the reference were to the number n. In fact, a great deal of the mistrust in the theory is due to this misunderstanding. The factor n in the expression np is always assumed to be large, although we do not know enough of the composition of np to be able to separate it into its factors.

It has been suggested that a better name would be the law of small probabilities. It is no doubt a law of small probabilities, because the probabilities considered are always small. If we want a short title, this would no doubt be better and save confusion. But if we want a title to give a complete description, this one does not tell the whole story. Bortkewitsch insists that the actual magnitude of the observed numbers is an important factor and that the main fault with the phrase, law of small numbers, lies in the fact that it emphasizes the actual observed numbers and not the corresponding small probabilities. In the limiting step leading to the derivation of the fundamental formula of small numbers, Bortkewitsch stresses the fact that p approaches zero and at the same time np and x should not be large. The number n is the only one in the formula which is large. Other names that have been suggested are "law of probability of rare events," "law of the permanence of small numbers," and one suggested recently by von Mises, "law of large numbers in the case of small frequencies." This latter name seems to be truly descriptive.

The attempt to cloud Bortkewitsch's title as originator of the law is rather futile. No doubt Poisson was the first to derive the underlying formula, but Bortkewitsch was the first to use the name and the first to point out the statistical importance of Poisson's formula. Poisson was the inventor of the phrase "law of large numbers," but he did not understand by it this or that formula from the theory of probabilities, he meant the characteristic behavior of the frequencies and the realization in experience.

The objection has been made that the use of the law of small numbers should be limited to cases where a test can be made to find out whether p is actually small and n large. The test is to find the fitted binomial. Even if p and n do satisfy these conditions the binomial will give just as good or a better fit. When the data in the problems in the literature are represented by binomials, all sorts of values of p and n are found. From this the objectors conclude that the application of the law of small numbers is not justified.

In regard to this objection Bortkewitsch points out that the *a priori* assumption of representation by a binomial with small p and large positive n carries with it the assumption that q is less than 1. It can easily be shown that the Lexis criterium of dispersion Q is equal to $\sqrt{q}$ for the binomial. Hence the assumption is the same as assuming a Q less than unity. This leads to the paradox that in an investigation of a series of small frequencies with regard to the question of dispersion, the assumption is made that subnormal dispersion is present. In other words, Bortkewitsch accepts the Lexis theory of dispersion as a measure of stability and his opponents reject it.

The binomial no doubt makes a better fit in many cases. In using the exponential form one uses only the mean of the observations and postulates absolute ignorance as to the deviations from the mean. On the other hand, in the binomial with its two constants at our disposal we can use the standard deviation of the observations about the mean. But part of the problem is the representation of the frequencies of a series of observations by the simplest possible mathematical expression. The dubious case of the binomial with negative n and p can be naturally explained from the standpoint of the Lexis theory. They simply express the fact that in the case in question the divergence coefficient is greater than unity, which may be considered as accidental or a mark of supernormal dispersion. It is not necessary that we consider the Poisson formula as coming out of the binomial expansion. In the semi-invariant theory of frequency functions the Poisson exponential follows from the special case in which the semi-invariants are all equal.

The law of small numbers is so closely tied up with the Lexis theory that they stand or fall together. Keynes suspects that the apparent paradox in the statement that the regularity of the occurrence of rare events is more stable than that of commoner events, may be explained by the peculiar measure of

stability which has been selected. There is considerable truth in this statement, nevertheless there is much common sense in the Lexis idea. The man of the street is not much concerned with formal definitions of stability, but he does think it a rather interesting fact that the average number of triple plays in one of our national base ball leagues is 4 and the chances are more than even that the number of triple plays in a season will not differ from 4 by more than 1. There may be more general and more accurate ways of describing the frequency of rare events, but experience has shown that the law of small numbers gives a simple and useful mathematical description.

A NEW FORMULA FOR VOLUME

By J. B. REYNOLDS, Lehigh University

One of the most remarkable theorems known to the Greeks is that for th volume generated when a plane area is revolved about a line. The theorem as stated by Pappus, who lived about the end of the third century A. D., is as follows:

Figures generated by a complete revolution of a plane figure about an axis are in a ratio compounded of (1) the ratio of the areas of the figures and (2) of the ratio of the straight lines similarily drawn to (i.e. drawn to meet at the same angles) the axes of rotation of the respective centres of gravity. Figures generated by incomplete revolutions are in the ratio compounded (1) of the ratio of the areas of the figures and (2) of the ratio of the arcs described by the centres of gravity of the respective figures, the latter ratio being itself compounded (a) of the ratio of the of the straight lines similarily drawn (from the respective centres of gravity to the axes of revolution) and (b) of the ratio of the angles contained (i.e. described) about the axes of revolution by the extremities of said lines (i.e. the centres of gravity).[1]

The theorem was stated without proof by Pappus in Book VII of his Mathematical Collections with the suggestion that proof would be given in Book XII which, if it ever existed, is not now extant. The theorem was apparently lost to be rediscovered, without satisfactory proof, by Guldin (1577–1643), a Swiss mathematician, whose name it often bears. Edwards in his recent work on integral calculus[2] shows that the theorem as stated by Pappus holds when the line of revolution is in the plane of the area revolved and does not cut the area (the usual interpretation) and also when the line of revolution is parallel to the plane of the area, or oblique to it, provided the projection of the line upon

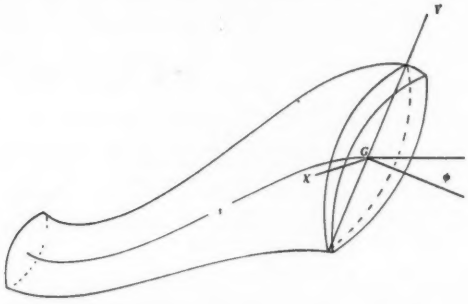
---

[1] See Sir Thomas Heath, *A History of Greek Mathematics*, vol. 2, p. 403.

[2] See Joseph Edwards, *Integral Calculus*, vol. 1, p. 783.

the plane of the area does not cut the area. Routh[1] drew attention to the fact that the line of revolution need not be fixed but may be the instantaneous axis of the moving area. From this point of view the volume generated equals the area times the length of path of its centroid.

It is the purpose of this article to develop a formula for volume for which the above are special cases. When a body may be considered as composed of a succession of laminas such that no lamina has faces that intersect as they approach coincidence and such that the face of lamina never becomes reentrant upon the volume, the volume may be considered as follows.



Let s be the space curve which is the locus of the centers of gravity of the laminas composing the body. Consider the volume of a differential lamina. Let the plane of one face be obtained from that of the other by rotation through an angle $d\theta$ about an instantaneous axis outside the face. Through the center of gravity G of one face draw an x-axis GX, parallel to the instantaneous axis. Let GY be a y-axis perpendicular to GX in the plane of the face, in which G lies, drawn away from the instantaneous axis. If the thickness of the lamina at G is dt, at the point (x, y) its thickness is $dt+yd\theta$. Hence, V being the volume, we have, to first order differentials in x, y, t and θ.

$$V = \iiint dx\,dy\,dt + \iiint y\,dx\,dy\,d\theta.$$

Since θ is independent of x and y, and since $\iint y dx dy = 0$ because G is the center of gravity of the face,

$$V = \iiint dx\,dy\,dt. \tag{1}$$

If s makes an angle ϕ with the normal to the plane, $dt = ds \cos \phi$ and[2]

[1] See E. J. Routh, *Analytical Statics*, vol. 1, p. 293.

[2] In the case where $\phi = 0$, GX lies in the binormal to the curve s and GY in the principal normal and $Rd\phi = ds$, where R is the radius of curvature of the curve s.

$$V = \iiint dx\,dy\cos\theta\,ds. \tag{2}$$

Supposing the integration in x and y to have been made for a general position of the lamina of face area A, we have

$$V = \int A\cos\theta\,ds, \tag{3}$$

in which A and ϕ are, in general, functions of s.

There are many special cases, some of which will be mentioned. If A is constant and $\phi = 0$, $V = As$, the case mentioned by Routh.

If A is constant; ϕ, constant, $= B$, and s is the circumference of a circle of radius r, $V = 2\pi rA\cos B$, the equivalent of special cases discussed by Edwards. If $B = 0$, $V = 2\pi r\,A$, the usual theorem of Pappus for volume. If the normal to A retains a fixed direction OX, then $\cos\phi ds = dx$ and we have the usual formula for volume by parallel sections. If the normal to A retains a fixed direction, s is a straight line, and A varies as s^2, we have a pyramid or cone; and if A is a constant, a cylinder.

A formula similar to (3),

$$S = \int L\cos\phi\,ds, \tag{4}$$

can easily be developed for the surface of the volume generated. L, being the perimeter of the face of a general lamina, is, in general, a function of s, the locus of the center of gravity of the perimeter.

A few applications may make clear some uses of the formulas. To find the volume of a truncated cylinder standing on a base of area A_0, we may consider it to be generated by a general lamina making a variable angle ϕ with the base which coincides with the base in its first position and with the truncated base in its final position. Then $A = A_0\sec\phi$, and $V = \int A\cos\phi ds = A_0 s$, the area of the base times the height at the center of gravity.

Suppose a cornucopia is generated by a variable circle whose plane moves normal to the helix $x = 4a\cos\theta$, $y = 4a\sin\theta$, $z = 3a\theta$, its center being on the helix and its radius equal to $s/8$ where s is the length of the arc of the helix measured from $\theta = 0$. If θ varies from 0 to $3\pi/2$, find the volume and surface of the cornucopia.

Here $A = \pi s^2/64$; $\phi = 0$, and $V = (\pi/64)\int s^2 ds$, from $s = 0$ to $s = 15\pi a/2$. Hence $V = 1125\pi^4 a^3/512$, and $S = \int L\cos\phi ds = (\pi/4)\int s ds$, from $s = 0$ to $s = 15\pi a/2$; whence $S = 225\pi^3 a^2/32$.

Again, consider the figure generated by a circle of constant radius r $(r < 2a/3)$ moving with its plane perpendicular to the plane of a fixed circle of radius a,

the center of the moving circle being always on the circumference of the fixed circle and the normal to the plane of the moving circle making an angle $\phi=\theta/2$ with the tangent to the fixed circle where it pierces the plane of the moving circle; to find the surface and volume generated for θ varying from 0 to π where $a\theta$ is the distance moved by the center of the generating circle.

In this example, $\phi=\theta/2$, $s=a\theta$ and hence $S=\int L\cos\phi ds=2\pi ra\int_0^\pi\cos(\theta/2)d\theta=4\pi ra$; and $V=\int A\cos\phi\, ds=\pi r^2a\int_0^\pi\cos(\theta/2)d\theta=2\pi r^2a$.

Many possible applications of the formulas will readily occur to anyone interested.

NOTE ON THE GEOMETRIC DESCRIPTION OF LINEAR FAMILIES OF CONICS BY MEANS OF APOLARITY

By ALAN D. CAMPBELL, Syracuse University

It is easy to describe geometrically one-parameter linear families (pencils) of point or line conics. For example the pencil of point conics $\lambda x^2+\mu(y^2+2zx)=0$, with discriminant $\Delta\equiv-\mu^3$, consists of all the point conics that intersect in four coincident points at (0, 0, 1) and hence has just one degenerate conic, a double line (the common tangent at this point). The pencil of line conics $\lambda(u^2-v^2)+\mu(u^2-w^2)=0$ consists of all the line conics with four common tangents and therefore has three degenerate conics, the pairs of opposite vertices of the complete quadrilateral formed by these four tangents.

To describe two-parameter linear families (nets) of conics and to derive as well as describe typical three- and four-parameter families we can use a projectively invariant property that links point and line conics and is called apolarity.[1] A point conic

$$C\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is said to be apolar to line conic

$$C'\equiv A'u^2+B'v^2+C'w^2+2F'vw+2G'wu+2H'uv=0$$

if

$$K=aA'+bB'+cC'+2fF'+2gG'+2hH'=0.$$

The following facts about apolarity are well known. If a line conic C' is a double point, then C' lies on any apolar point conic C; if C' is a pair of points, then it is a pair of conjugate points with respect to any apolar point conic C; if C' is a nondegenerate line conic, then it is inscribed in complete quadrilaterals

[1] See G. Salmon, *Conic Sections*, 2nd edition, §375; H. J. S. Smith, *Collected Works*, vol. 2, pp. 524–540; Reye-Holgate, *Geometry of Position* pp. 222–240; and, for the space analogue, Snyder and Sisam, *Analytic Geometry of Space*, pp. 180–187. We note that apolarity is an invariant property in the complex domain, in the real domain, and in the Galois Fields of order p^n where $p>2$.

that are self-polar with respect to any apolar point conic C (i.e., the pairs of opposite vertices of such a quadrilateral are pairs of conjugate points with respect to C).

By an r-parameter linear family ($r=0, 1, 2, 3, 4, 5$) of point (or line) conics, we mean all conics given by the equation

$$\lambda_1 C_1 + \lambda_2 C_2 + \cdots + \lambda_{r+1} C_{r+1} = 0, \text{ where } C_i = 0 (i = 1, 2, \cdots, r+1)$$

are linearly independent point (line) conics that are called fundamental conics of the family. If a point (line) conic is apolar to each of a set of fundamental line (point) conics of an r-parameter family, it is apolar to every conic of this family. Also if a set of fundamental conics of a family of point (line) conics are apolar to the conics of another family of line (point) conics, then every conic of the first family is apolar to every conic of the second. Two such families are said to be apolar to one another. To every r-parameter family $M_r (r=0, 1, 2, 3, 4)$ of line conics there corresponds a unique apolar $(4-r)$-parameter family N_{4-r} of point conics, and conversely. Hence if we project any family M_r into another family M_r' of the same class of families (for which M_r may be taken as a typical family), then (since apolarity is an invariant property) the corresponding unique apolar family N_{4-r} must go into the unique family N_{4-r}' apolar to M_r'; and conversely. So to the class of families of line conics to which M_r belongs there corresponds a unique class of apolar families of point conics to which N_{4-r} belongs and conversely. We call two such classes apolar classes of families.

Thus apolarity gives us a very useful one-to-one reciprocal correspondence between the classes of line (point) conics and their apolar classes of four-parameter families of point (line) conics, between the classes of pencils of line (point) conics and their apolar classes of three-parameter families of point (line) conics, and between the classes of nets of line (point) conics and their apolar classes of nets of point (line) conics. To describe geometrically a family of point conics apolar to a family M_r of line conics, we choose $r+1$ linearly independent conics in M_r, preferably double points (if these are present in the family), otherwise pairs of real points or, finally, pairs of conjugate imaginary points, or nondegenerate real or imaginary conics.

We shall now give a few illustrations of this use of apolarity, first of all in the study of three- and four-parameter families of conics. It is easy to see that, apolar to the double point $u^2=0$, we have the four-parameter family

$$\lambda y^2 + \mu z^2 + 2\nu yz + 2\rho zx + 2\sigma xy = 0,$$

which consists therefore of all the conics through a given point $P(1, 0, 0)$. Apolar to the typical pencil $\lambda(u^2-v^2)+\mu(u^2-w^2)=0$, we have (using K) the three-parameter family

$$\lambda(x^2 + y^2 + z^2) + 2\mu yz + 2\nu zx + 2\rho xy = 0,$$

which consists of all the conics having a common self-polar quadrilateral.

The typical nets of point (line) conics are derived by other methods than apolarity,[1] but they are readily described by means of their apolar nets of line conics. Let us consider the net of point conics

$$\lambda(z^2 + 2xy) + 2\mu yz + \nu x^2 = 0.$$

Any line conic apolar to the fundamental conics of this net must satisfy the equations (see K): $C'+2H'=0$, $F'=0$, $A'=0$. So every line conic of the form, $B'v^2+C'w^2+2G'wu-C'uv=0$ is apolar to this net. We put $C'=\lambda$, $G'=\mu$, $B'=\nu$, and we have an apolar net of line conics

$$\lambda(w^2 - uv) + 2\mu uw + \nu v^2 = 0.$$

In this net we find a complete quadrilateral with vertices given by $uw=0$, $(w-v)(w+v+u)=0$ (the conic given by $\lambda=1$, $\mu=\frac{1}{2}$, $\nu=-1$) and $(w+v)(w-v-u)=0$ (the conic given by $\lambda=1$, $\mu=-\frac{1}{2}$, $\nu=-1$) and also the point $v^2=0$ on one of the sides of this quadrilateral. Hence any net of point conics of the class that has the above typical net consists of all the conics having a common real self-polar quadrilateral and also passing through a given point on one of its sides. The net of point conics

$\lambda x^2 + \mu y^2 + 2\nu yz = 0$ is apolar to the net of line conics $\lambda w^2 + 2\mu wu + 2\nu uv = 0$;

hence this net of point conics consists of all the conics that have a given point $(1, 0, 0)$ and a given line $x=0$ as pole and polar and that cut $x=0$ in the given point $(0, 1, 0)$.

---

[1] For the complex domain, see C. Jordan, *Réduction d'un réseau de formes quadratiques*, Journal de Mathématiques (6), vol. 2 (1906), p. 412. For the Galois fields of order p^n where $p>2$, see A. H. Wilson, *The canonical types of nets of modular conics*, American Journal of Mathematics, vol. 36 (1914), pp. 187–210. For the real domain, see A. D. Campbell, *Nets of conics in the real domain*, American Journal of Mathematics (not yet published).

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. AN ANALYTICAL DISCUSSION OF PROBLEM 3221

BY ROSCOE WOODS, University of Iowa

In the October, 1927 issue of this Monthly there appeared my geometric solution of Problem 3221 set by H. E. Trefethen. The problem read as follows: A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

Let us consider the problem somewhat extended, using analytical methods. Let the vertices of the variable rectangle be O, A, B, C, where O is the fixed vertex, A and C are the vertices on the two fixed perpendicular straight lines, and B is the remaining vertex whose locus is evidently a circle with O as center and radius r which is the length of the constant diagonal of the variable rectangle. There is no loss of generality if the units are so chosen that $r=1$. Let the vertices O, A, B, C have the coordinates $(0, 0)$, $(X, 0)$, (X, Y) and $(0, Y)$ respectively where $X^2+Y^2=1$. Instead of finding the locus of $P(x, y)$, the foot of the perpendicular from B upon AC, let the locus of P, the foot of the perpendicular from $B'(X', Y')$ upon AC, be discussed, where X and Y are related to X' and Y' as follows:

$$(1)\qquad \begin{aligned} X' &= (a_1X + b_1Y + c_1)/(a_3X + b_3Y + c_3), \\ Y' &= (a_2X + b_2Y + c_2)/(a_3X + b_3Y + c_3), \end{aligned}$$

where a_3, b_3, c_3 are not all zero simultaneously and where a_i, b_i, $c_i (i=1, 2, 3)$ are any real numbers including zero. The coordinates of $P(x, y)$ are

$$(2)\qquad x = X(XX' - YY' + Y^2), \qquad y = Y(YY' - XX' + X^2).$$

For convenience of discussion we introduce a parameter t as follows:

$$X = (1 - t^2)/(1 + t^2), \quad Y = 2t/(1 - t^2).$$

It then follows that

$$(3)\qquad \begin{aligned} X' &= [t^2(c_1 - a_1) + 2b_1t + a_1 + c_1]/[t^2(c_3 - a_3) + 2b_3t + a_3 + c_3], \\ Y' &= [t^2(c_2 - a_2) + 2b_2t + a_2 + c_2]/[t^2(c_3 - a_3) + 2b_3t + a_3 + c_3]. \end{aligned}$$

From (2) and (3) it now follows that the locus of P is in general a rational octavic curve. This locus has cusps at the circular points of the plane and cuts the line at infinity in two other points which are coincident when $b_3^2+a_3^2=c_3^2$. It is also easy to see that this locus cuts the x- and y-axes in the points

$$[c_1 \pm a_1)/(c_3 \pm a_3), \quad 0] \quad \text{and} \quad [0, \ (c_2 \pm b_2)/(c_3 \pm b_3].$$

Let the determinant of the transformation (1) be denoted by D. If D is different from zero, (2) is an eight parameter system of rational curves. For $D \neq 0$, P' lies upon a conic, the transform of the circle $X^2+Y^2=1$ under (1). There are many interesting curves in this system when the arbitrary constants are subjected to various conditions. For instance, when (1) reduces to the identity, the locus of P is the hypocycloid of four cusps. As is known this curve is a sextic with cusps at the circular points.

Also there are septimics, sextics, quartics, ellipses, and a circle. There are no quintics or cubics. In what follows are set forth some of the conditions that give rise to these curves.

(I) Septimics with cusps at the circular points arise when $\pm a_3 = \mp c_3 = b_3 = k/2$, where k is any constant different from zero. In this case $D \neq 0$ and the unit circle is transformed into a hyperbola, the discriminant of the second degree terms being proportional to D^2. Septimics can also arise when $a_i = \pm c_i (i = 1, 2, 3)$. In this case $D = 0$ and the locus of P' is the line whose equation is

$$(a_3b_2)X' + (a_1b_3)Y' + (a_2b_1) = 0 \qquad \text{where} \qquad (a_ib_k) = a_ib_k - a_kb_i.$$

(II) Sextics with cusps at the circular points arise: (a) when $a_3 = b_3 = 0$, $c_3 = k (k \neq 0)$, and $D \neq 0$. If we impose the further conditions $a_2 = c_2 = b_1 = c_1 = 0$ and $k = 1$, the system so obtained is symmetrical to the x- and y-axes and the origin and is of such a nature that a great deal of information is readily obtained. We shall return to this system later. (b) when $c_3 = b_3 = c_2 = b_2 = 0$, $a_3 = a_2 = k$ $(k \neq 0)$ and $D = 0$. In this case P' lies on the line $Y' = k$. (c) when the rank of D is 1. In this case P' is now a fixed point in the plane and the locus so found is the pedal curve of this point with respect to the hypocycloid of four cusps. If P' lies inside the hypocycloid the pedal has four real branches passing through P', and if on the outside of the hypocycloid there are only two real branches passing through P'. In particular if P' is at the origin the pedal is the four-leaved rose.

(III) When $a_1+b_2=k$, $b_1-a_2=0$, $a_3=b_3=0$, $c_3=k$ $(k \neq 0)$ and $D \neq 0$, the locus becomes bi-circular quartics with double points at the circular points. The locus of P' is now a parabola. The x- and y-intercepts of these curves are easily found but are not set down here.

(IV) In (III) if the further conditions are imposed, that is if $c_2=c_1=0$, $D=0$, the bi-circular quartics degenerate into ellipses. In this case the locus of P' is an ellipse. If we set $2a_1=k$, the locus becomes a circle. It ought to be pointed out that in this case (IV) P' is a fixed point on the line AC.

Returning now to the system of sextics in (IIa), we note several cases of interest:

(1) When a_1, $b_2>1$, $a_1+b_2>1$, the sextics cross the axes at A_1 $[\pm a_1, 0]$, $B_1[0, \pm b_2]$, $A_2[\pm b_2^{1/2}(a_1+b_2-1)^{-1/2}, 0]$, and $B_2[0, \pm a_1^{1/2}(a_1+b_2-1)^{-1/2}]$. The points A_2 and B_2 are double points.

(2) When $0<a_1<1$, $0<b_2<1$, the curves have intercepts at A_1 and B_1 but the points A_2 and B_2 are imaginary.

(3) When a_1, $b_2<0$, the curves cross at A_1 and B_1 and have double points at A_2 and B_2 and there is also a double point in each quadrant.

(4) When $a_1=1$, $b_2>0$, $(b_2\neq 1)$ the points A_1 and A_2 coincide and become cusps, B_1 remain intercepts and B_2 remain double points.

(5) When $a_1=1$, $b_2=0$, $D=0$, the locus is now the diameter of the unit circle that lies along the x-axis.

(6) When $a_1=1$, $b_2<0$, A_1 and A_2 coincide and are cusps, B_1 are intercepts and B_2 are imaginary.

(7) When $a_1>1$ or $a_1<1$, $b_2=0$, $D=0$, the locus has A_1 for intercepts, B_1 and A_2 are at the origin, and B_2 remain double points.

(8) When $0<a_1<1$, $b_2=0$, $D=0$ the locus has intercepts at A_1, B_1 and A_2 fall at the origin and the points B_2 are imaginary.

(9) The interchange of the conditions imposed upon a_1 and b_2 merely has the effect of interchanging the x- and y-axes in the foregoing.

(10) When $a_1=b_2=1$, the locus is the hypocycloid of four cusps, A_1 coinciding with A_2, B_1 with B_2, forming cusps.

II. A Method of Solving A Biquadratic

By C. H. Smiley, University of Illinois

The problem of finding the four roots of the biquadratic

$$f(x) \equiv a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \qquad (a_0 \neq 0)$$

may be considered as equivalent to the problem of factoring $f(x)$ into two quadratic factors; that is, the determination of b_0, b_1, b_2; c_0, c_1, c_2 so that $f(x) \equiv (b_0x^2+b_1x+b_2)(c_0x^2+c_1x+c_2)$.

We must then determine these six quantities to satisfy the five equations:

$$a_0 = b_0c_0\ ;\ a_1 = b_0c_1 + b_1c_0\ ;\ a_2 = b_0c_2 + b_2c_0 + b_1c_1\ ;\ a_3 = b_1c_2 + b_2c_1\ ;\ a_4 = b_2c_2.$$

Having one condition at our disposal, let us set $b_1=c_1$. (We shall discuss

the legitimacy of this assumption later.) Then our conditions may be written:

(1) $b_1=c_1$; $b_0c_0=a_0$; $b_0+c_0=a_1b_1^{-1}$; $b_0c_2+b_2c_0+b_1^2-a_2=0$; $b_2+c_2=a_3b_1^{-1}$; $b_2c_2=a_4$.

Then b_0 and c_0 are the roots of the quadratic equation, $y^2-a_1b_1^{-1}y+a_0=0$.

And likewise, b_2 and c_2 are the roots of the quadratic equation, $z^2-a_3b_1^{-1}z+a_4=0$.

Then, for an assumed b_1, we can compute the roots of these two quadratics, which will be the corresponding values of b_0, c_0, b_2, c_2. Using these values in the fourth equation of (1), we may obtain a better value of b_1, and by a method of successive approximations determine a set of constants which will satisfy the fourth equation of (1) to any desired degree of accuracy. Or we may use a method of interpolation to secure this same result.

Having determined a set of constants which will satisfy the conditions (1), we may proceed to solve the two quadratics to obtain the four roots of the biquadratic.

It is interesting to note the nature of b_1, when expressed as a function of the roots of the biquadratic. For the sake of simplicity, let us set $a_0=1$ and note that the fourth equation of (1) may be written in the form:

$$b_1^2 = a_2 - \frac{b_2}{b_0} - \frac{c_2}{c_0}\,; \quad b_1^2 = x_1x_3 + x_2x_4 + x_1x_4 + x_2x_3\,;$$

the corresponding resolvent cubic is

$$b_1^6 - 2a_2b_1^4 + (a_2^2 + a_1a_3 - 4a_4)b_1^2 + a_1^2a_4 + a_3^2 - a_1a_2a_3 = 0.$$

Now as to the legitimacy of our assumption, $b_1=c_1$, obviously there are two cases which we will need to avoid;—first, the case where the only possible factorization into quadratic factors with real coefficients leaves one factor with and the other factor without a term in x; and secondly, where the only possible factorizations into quadratic factors with real coefficients leave the two coefficients of x in each pair of factors, of opposite sign.[1] The first case may be handled by the method of this paper if we first transform our equation by means of a simple translation of the origin. And the second case may also be handled if we change the signs of all the a's, making a_0 negative, before starting our approximations. There may be some difficulty in recognizing these cases—and it seems desirable to avoid them entirely. This we may do, if we first transform our given biquadratic in such a way as to make a_1 zero, and then change the signs of all the a's to make a_0 negative. Then the method will work.

---

[1] We have tacitly assumed that $a_0>0$ in the given biquadratic.

III. On the Approximate Division of a Circumference

By Tobias Dantzig, University of Maryland

In a recent issue of this Monthly,[1] R. A. Johnson discussed two approximate constructions for the division of a circle in n equal parts. While the accuracy of either construction, particularly the second one, was reasonably good, the error did not approach zero with increasing n. The approximation method given below is also rather simple and accurate, but has the advantage that the error very rapidly approaches zero with increasing n.

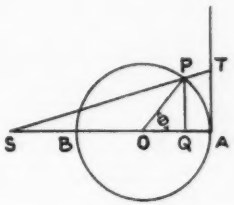


Fig. 1

Let AB (Fig. 1) be the diameter of a unit circle, S a point on the extension of this diameter, and P the representative point of an acute angle θ. Let SP continued meet the tangent at A in T and set $AT=t$, $BS=c$. Then by similar triangles:

$$\frac{AT}{QP}=\frac{AS}{QS} \text{ or } t=\frac{(c+2)\sin\theta}{c+1+\cos\theta},$$

and consequently

$$t-\theta=\frac{(2+c)\sin\theta-(1+c)\theta-\theta\cos\theta}{1+c+\cos\theta}.$$

Expanding the numerator into series, we obtain

$$t-\theta=\frac{(1-c)\theta^3}{6(1+c+\cos\theta)}+F(\theta)\cdot\theta^5,$$

where $F(\theta)$ is a function of θ such that

$$\lim_{\theta\to 0}F(\theta)=(c-3)(c+2)^{-1}/120.$$

Thus it appears that $t-\theta$ is, for any value of c, an infinitesimal of the third order and *for $c=1$ the difference $t-\theta$ becomes an infinitesimal of the fifth order with respect to θ*. This gives the approximation formula

$$\theta=\frac{3\sin\theta}{(2+\cos\theta)}. \tag{1}$$

The accuracy of the formula is shown in the following table:

---

[1] Vol. 34 (1927), pp. 429–431.

n	Angle	Arc	Approximation	Error in percentage
5	72°	1.2566	1.2360	1.6
6	60°	1.0472	1.0392	0.8
12	30°	.5236	.5234	0.4
30	12°	.2094	.2094	0.005

This leads to the approximate division shown in Fig. 2. On the tangent CD lay off a uniform scale with a unit equal to $R/6$. Let N be the point on this scale representing the number n of parts in which we wish to divide the circumference. Let NO meet AD in T and let ST meet the first quadrant in P_n; then the arc AP_n is approximately $1/n$ of the circumference.

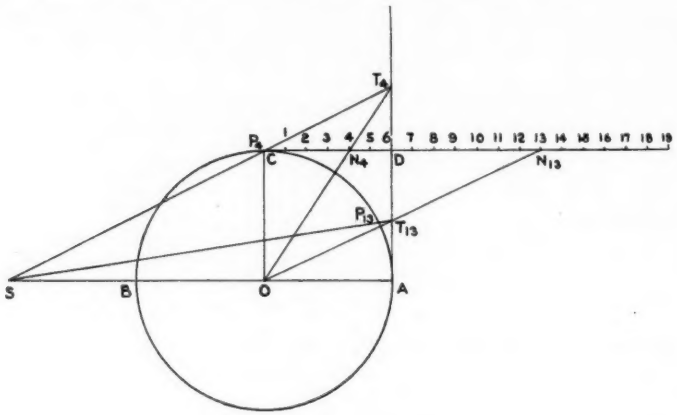


Fig. 2

Incidentally, formula (1) for $\theta = \pi/6$ gives the following approximation for π: $\pi = 18/(4+\sqrt{3})$, which is correct to two decimal places.

Figure 2 shows the construction for $n = 4$ and $n = 13$.

While the construction here given is believed to be new, formula (1) on which it is based dates back to Cardinal Cusanus (1401). It was later derived independently by Willebrord Snellius (1627), who also gave another approximation formula $\theta = \frac{1}{3}(2 \sin \theta + \tan \theta)$. It is of considerable historical interest to note that while Snellius regarded the formulae as mere approximations, Cusanus actually believed that he was offering an exact construction. As a matter of fact he utilized this method for a determination of the number π.

In this connection we wish to note two other formulae for the approximation of an arc of a circle:

(2) $\theta = (\sin^2\theta \tan\theta)^{1/3}$; (3) $\theta = \dfrac{(14 + \cos\theta)\sin\theta}{(9 + 6\cos\theta)}$

The first one is due to Orontius Finaeus whereas the other was devised by Newton and gives an exceedingly accurate construction. As a matter of fact Newton used the formula for getting the number π correct to 12 decimal places.

For references to the problem of division of the circumference and allied questions we refer the reader to Moritz Cantor, *Geschichte der Mathematischen Wissenschaften*, volume 2 and to Theodor Vahlen, *Konstruktionen und Approximationen*.

IV. A Problem in Maxima and Minima

By Roger A. Johnson, Hunter College of the City of New York

Problem. *A frustum of a right circular cone with a fixed altitude h and a fixed lower base of radius r has a variable upper base of radius x. Consider the area of the lateral surface as a function of x in the interval from $x=0$ (cone) to $x=r$ (cylinder). In particular, for what values of x in this interval is the area a maximum or minimum?*

This problem is noteworthy for several reasons. It is suggested by a standard problem of the calculus of variations, that of the surface of revolution of minimum area.[1] It is a typical problem in maxima and minima, of the class ordinarily solved in a first course in the calculus, but the author has not seen it proposed in any calculus text. Finally, the nature of the solution is by no means intuitionally evident.

The reader is invited at this point to consider the problem, and before turning the page to formulate intuitionally his own conclusions as to its solution.

The straightforward solution of the problem by the usual methods of the calculus should not tax the powers of the average undergraduate. The area in question is

(1) $$S = \pi(r + x)\left[h^2 + (r - x)^2\right]^{1/2},$$

the last factor representing the slant height g. In particular, the areas of the cylinder ($x=r$) and of the cone ($x=0$) are, respectively,

$$S_c = 2\pi rh, \quad S_0 = \pi r\left[h^2 + r^2\right]^{1/2}.$$

[1] For if we consider the surfaces generated by revolving about an axis the various curves joining two fixed points A and B, where AB is parallel to the axis, we naturally think first of the cylinder generated by AB itself, and secondly of the double frustum generated by a broken line ACB, with $AC=CB$.

Differentiating (1), we have

$$(2)\qquad \frac{dS}{dx} = \pi\left[g - \frac{(r+x)(r-x)}{g}\right] = \pi\frac{g^2-(r^2-x^2)}{g} = \frac{\pi(x^2-rx+\frac{1}{2}h^2)}{2g}.$$

Now this derivative is positive when $x=0$, and also when $x=r$, thus tempting the unwary to the inference that S increases monotonically from $x=0$ to $x=r$, and that within this interval there is no maximum nor minimum. A second thought, however, to the effect that when h is small as compared with r the area of the cylinder is less than that of the cone, impresses on us that the situation is not so simple. We therefore proceed to transform the derivative,

$$(3)\qquad \frac{ds}{dx} = \frac{\pi}{2g}\left[\left(x-\frac{r}{2}\right)^2 + \frac{1}{4}(2h^2-r^2)\right]$$

from which it appears that if $2h^2>r^2$, the derivative is always positive and S does indeed increase monotonically as the frustum changes from conical to cylindrical form. But if $r^2>2h^2$, we have the surprising result that the derivative vanishes twice between $x=0$ and $x=r$; in other words, under these conditions, the area has both a maximum and a minimum within the interval. As the upper base of the frustum increases, the lateral area first increases to a maximum, then decreases to a minimum, then increases to that of the cylinder. The maximum and the minimum are given by values of x equidistant from $r/2$:

$$x_1 = \tfrac{1}{2}(r-(r^2-2h^2)^{1/2}),\quad x_2 = \tfrac{1}{2}(r+(r^2-2h^2)^{1/2}).$$

It is of interest to inquire further whether the relative maximum at x_1, or the cylinder S_c, is greater; and similarly to compare the relative minimum at x_2 with the cone S_0. Without much difficulty the following results are established:

I. If $r<h\sqrt{2}$, the area increases monotonically as x increases from 0 to r. If $r=h\sqrt{2}$, the area is stationary when $x=\frac{1}{2}r$.

II. If $r>h\sqrt{2}$, the area increases from S_0 to a maximum S_1 at x_1, then decreases to a minimum S_2 at x_2, then increases to the final value S_c when $x=r$.

(a) The area of the cone S_0 is less than that of the relative minimum frustum at x_2, provided that $r/h<\frac{1}{2}\sqrt{(5+3\sqrt{3})}=1.596$. If the ratio of r to h exceeds this value, then x_2 gives the minimum area in the whole interval.

(b) Similarly the cylinder S_c is greater than the relative maximum at x_1, provided $r/h<\frac{1}{8}\sqrt{(11+5\sqrt{5})}=1.665$; but when the ratio exceeds this value, x_1 furnishes an absolute maximum.

(c) When r/h exceeds $\sqrt{3}$, the area of the cone S_0 exceeds that of the cylinder S_c.

V. On the Integration of a Certain Type of First Order Differential Equation

By John P. Dalton, University of the Witwatersrand, Johannesburg, S.A.

1. Problem 3272 recently proposed in this Monthly[1] by L. M. Berkeley is readily solved by either of the following methods. These methods have been used for some time time past by the writer in his lectures to undergraduates; and although the problem proposed for solution is of little intrinsic interest, this opportunity is taken of making the methods known.

2. The type of equation considered is

$$(P_1 + P_2)dx + (Q_1 + Q_2)dy = 0, \tag{1}$$

where P_1, P_2, Q_1 and Q_2 are functions of x and y, continuous in both variables in the domain under consideration.

The first method depends upon the possibility of obtaining an integrating factor common to both differentials P_1dx+Q_1dy and P_2dx+Q_2dy.

If possible let μ be such a factor and λ its logarithm. Further let

$$D_r = \frac{\partial P_r}{\partial y} - \frac{\partial Q_r}{\partial x} \quad (r = 1, 2), \qquad \Delta = \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} \neq 0,$$

$$\Delta_P = \begin{vmatrix} P_1 & D_1 \\ P_2 & D_2 \end{vmatrix}, \qquad \Delta_Q = \begin{vmatrix} Q_1 & D_1 \\ Q_2 & D_2 \end{vmatrix}.$$

On applying the Euler condition to the two differentials and solving the resulting equations for the derivatives of μ we obtain

$$\frac{\partial \lambda}{\partial x} = \frac{\Delta_P}{\Delta}, \tag{2}$$

$$\frac{\partial \lambda}{\partial y} = \frac{\Delta_Q}{\Delta}, \tag{3}$$

The commutative property then gives as the criterion for the existence of μ

$$\frac{\partial}{\partial y}\left(\frac{\Delta_P}{\Delta}\right) = \frac{\partial}{\partial x}\left(\frac{\Delta_Q}{\Delta}\right) \tag{4}$$

If this equation is satisfied, integration of (2) and (3) gives the required integrating factor.

3. The equation referred to in §1 is

$$(\sin y - m \tan x) \cos y\, dx - (\tan x + m \sin y)\, dy = 0, \tag{5}$$

where m is a constant. Writing

$$P_1 = \cos y \sin y, \quad P_2 = -m \tan x \cos y, \quad Q_1 = -\tan x, \quad Q_2 = -m \sin y,$$

---

[1] Vol. 34 (1927), p. 380.

we get

$$D_1 = \cos 2y + \sec^2 x, \qquad \Delta = -m \cos y(\sin^2 y + \tan^2 x),$$

$$D_2 = m \tan x \sin y, \qquad \Delta_P = m \tan x \cos y(\cos^2 y + \sec^2 x),$$

$$\Delta_Q = 2m \sin y \cos^2 y.$$

Hence

$$\frac{\partial}{\partial y}\left(\frac{\Delta_P}{\Delta}\right) = \frac{\sin 2x \sin 2y}{(1 - \cos^2 x \cos^2 y)^2} = \frac{\partial}{\partial x}\left(\frac{\Delta_Q}{\Delta}\right)$$

and condition (4) is satisfied.

On integrating

$$\frac{\partial \lambda}{\partial y} = \frac{\Delta_Q}{\Delta} = -\frac{\sin 2y}{(\sin^2 y + \tan^2 x)},$$

we obtain

$$\log \mu = -\log (\sin^2 y + \tan^2 x) + \log F(x),$$

where $F(x)$ is an arbitrary function

Substituting this in

$$\frac{\partial \lambda}{\partial x} = \frac{\Delta_P}{\Delta} = -\frac{\tan x(\cos^2 y + \sec^2 x)}{\sin^2 y + \tan^2 x},$$

we obtain $F(x) = \sec x$. Hence the desired integrating factor is

$$\mu = \frac{\cos x}{(1 - \cos^2 x \cos^2 y)}$$

and therefore the solution of equation (5) is

$$\arctan (\sin x \cot y) = \log C\left(\frac{1 - \cos x \cos y}{1 + \cos x \cos y}\right)^{m/2}.$$

4. The second method applicable to differentials of the type now under consideration is of a more tentative nature; nevertheless it is of some utility.

Let μ_1 be an integrating factor of the differential $P_1dx + Q_1dy$; and let $\zeta_1 = \text{constant}$ be the resulting integral. Let μ_2 and ζ_2 be similar functions relative to the differential $P_2dx + Q_2dy$.

Expressing μ_1 and μ_2 in terms of ζ_1 and ζ_2, we then determine arbitrary functions f and F such that

$$\mu_1 f(\zeta_1) = \mu_2 F(\zeta_2), \tag{6}$$

and this expressed in terms of x and y is an integrating factor for equation (1).

5. Applying this method to the problem under discussion we find, for the subdivisions of §3,

$$\mu_1 = \sec y \operatorname{cosec} y \cot x ; \qquad \zeta_1 = \log \sin x - \log \tan y ;$$
$$\mu_2 = \sec y ; \qquad \zeta_2 = \log \cos x + \log \cos y.$$

From these equations we find

$$\frac{\mu_2}{\mu_1} = -\frac{\sinh \zeta_2}{\cosh \zeta_1}$$

Hence from (6) $\mu_1 \operatorname{sech} \zeta_1$ is an integrating factor of (5) and this reduces to $\cos x[1-\cos^2 x \cos^2 y]^{-1}$ as before.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N.Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Mengenlehre. By F. HAUSDORFF. Berlin-Leipzig, Walter de Gruyter and Co., 1927. 285 pages.

Any attempt to produce a book on the theory of aggregates and have it approximately up to date and complete at the time of its appearance must, on account of the present status of the subject, fall somewhat short of the mark. Each year there appear one or two volumes of Fundamenta Mathematicae, a journal which is devoted almost entirely to this subject; and perhaps an equal volume of papers are published each year in German and American journals. When one considers the vast number of new results being obtained all the time, and realizes the practical impossibility of foretelling of what importance a new proposition is going to be when first discovered, one readily appreciates the difficulties which confront an author in culling out the new material and selecting the results which will likely prove most useful in future research. This is especially true on account of the comparative youth of the subject and the scanty supply of books available on this subject. However, in consideration of the difficulties, Professor Hausdorff seems to have accomplished this feat with a fair degree of success in his new book, that is, in so far as it was attempted. Such does not seem to have been the author's chief purpose in writing the book. This book appeals to the reviewer as one which would be of much more interest and practical value either to a beginner in the subject or to the mathematician

whose main field lies in some other branch of mathematics but who has occasional opportunity in his work to use some elementary theorems on aggregate theory than it would be to the research specialist in this particular field.

The reviewer would hardly feel justified in criticizing the author for his choice of the material to include in his book, because that is the author's pleasure; and undoubtedly the material chosen is fairly representative of the work which has been done and is being done in the field of aggregate theory, and it is capable of giving the beginner a nice insight into some of the beauties of the subject. However, there are a number of places in the book where the treatment could have been somewhat clarified and the proofs of some of the theorems materially shortened if only the author had chosen to make use of some quite well known theorems which were not included in his book. One or two instances of this sort will be given below.

We do not purpose, in this review, to give a synopsis of the material covered in the book under review, nor to compare it with an earlier book, *Grundzüge der Mengenlehre*, by the same author. The reader who desires this is referred to an excellent review of this book by H. M. Gehman in the Bulletin of the American Mathematical Society.[1] Instead, some features of the book which are of particular interest to the reviewer and which perhaps will be of interest to readers of the Monthly will be commented upon.

A very notable feature of this book is the author's attempt to introduce a logical system of notation and terminology. In most cases the author has followed the modern tendencies in this respect. The problem of adopting a standard system of notation and terminology is squarely up to workers in this field at the present time. Some symbolic language clearly is necessary in order to prevent our papers being cumbersome. However, the question as to just how far one should go in this respect is a debatable one. Especially is this true in point set theory. It is quite possible, in fact it often happens, that an author uses so many signs and symbols in his paper that his article is made almost unreadable. In the reviewer's opinion, letters and symbolic notation are of no use whatever in any subject unless they simplify the presentation and very materially shorten it. Yet it quite frequently occurs that an author of a paper in the field goes to considerable trouble to introduce a host of letters to mean certain things, and sets up his hypothesis and conclusions in the form of very imposing appearing equations when, as a matter of fact, the same thought could be conveyed more clearly and intelligibly in words and this could actually be done in much less space. This may appear rather astounding, but if one considers the extra space required in printing an equation or a number of sigma signs, which usually have to be set off, together with the space required to

[1] Vol. 33 (1927), pp. 778–780.

define these symbols, such a statement appears less unreasonable. The chief objection to a complicated system of notation, let me repeat, is the fact that the paper thereby becomes much more tedious to read. This objection can be largely overcome through the functioning of two agencies, to wit, (1) the adoption by all writers in the field of a standard set of symbols and terms to stand for the more commonly used notations whose expression in words either is awkward or requires considerable space, and (2) the exercise of great care by each writer when introducing a new symbol—the symbol should be suggestive, if possible, of the idea for which it stands, and it should be defined in each and every paper in which it is used until its usage by other authors becomes sufficient to warrant the omission of the definition. The reviewer knows of nothing which can do more to bring these agencies to bear in the field than the appearance of books like Hausdorff's in which a very elegant system of notation is used which follows, in the main, the modern tendencies in the subject. Hausdorff's symbols are all clearly defined and are, for the most part, quite suggestive of the things for which they stand. In some cases it seems to the reviewer that too much notation is used and several unnecessary symbols are introduced, but of course that is questionable. The notation $a \epsilon A$ to mean "a is an element of A" and $a \bar{\epsilon} A$ to mean "a is not an element of A" seems objectionable in some respects. Particularly it is true since, just two pages further on, the author introduces the notation $A \leqq B$ to mean "A is a subset of B", and $A < B$ to mean "A is a proper subset of B". It is quite true that $a \epsilon A$ and $a \leqq A$ do not convey exactly the same meaning, since the first indicates that a is one of the elements of A, while the second merely indicates that a belongs to A (a might be either a single element or a collection of elements of A); nevertheless, it seems that we should be able to do without the former of these symbols. In general usage it would ordinarily be known whether or not a consisted of a single element or a collection of elements; and such being known, the distinction between the notations $a \epsilon A$ and $a \leqq A$ appears to the reviewer to become unimportant. The notation $a \bar{\epsilon} A$ is objectionable for the additional reason that the notation $\overline{X}$ means the set X plus all of its limiting elements, and to use $\bar{\epsilon}$ in an entirely different sense is confusing to a reader even though he can soon grasp the correct meaning. Another confusing thing is the author's use of the term "Entfernung," which we would probably translate as "distance", for one thing and the term "Distanz" for a different thing. The former of these is denoted by xy (for distance from x to y) and the latter by $\overline{xy}$. On the whole, however, the system of notation is very good and is one which authors in the subject would do well to study and follow wherever it is necessary or desirable to resort to symbolic language. The use of the term *component* to replace the rather awkward term *maximal connected subset* seems especially desirable to the reviewer.

We consider now §39, 1 of the book, which is entitled *Bedingungen für einfache Kurven*. Here the author lists the following five conditions as necessary conditions in order that a metric space C be a simple continuous arc.

(α) C is self-compact.

(β) C contains two points a and b between which it is an irreducible continuum.

(γ) C contains two points a and b between which it is irreducibly connected.

(δ) [a condition equivalent to (γ)].

(ϵ) C is locally connected, i.e., connected im kleinen. He then proceeds to state (cf. Theorems III and IV) that the sets of conditions (α, γ), (α, δ), and (α, β, ϵ) are necessary and sufficient in order that C should be a simple continuous arc. No proof is given for the sufficiency of the sets (α, γ) and (α, δ); instead, references are made to N. J. Lennes and W. Sierpinski, respectively. However, the author gives a complete proof for the sufficiency of the set of conditions (α, β, ϵ). This proof is very elementary in nature, and we do not criticize the author in the least for giving it. It is interesting, however, to see how the theorem follows by the following line of argument: Since C is compact, by (α), and metric and, by (ϵ), is locally connected, it is readily seen that C satisfies axioms 1, 2, and 4 of R. L. Moore's paper *On the foundations of plane analysis situs*.[1] Hence by Theorem 15 of that paper, the proof of which uses essentially only these axioms, it follows that C is arcwise connected. But, by (β), C is an irreducible continuum between some two of its points a and b; and since C contains an arc from a to b, it is clear that C must be identically this arc.

It is of interest in this connection to note that if C is a subset of a Euclidean space of any number of dimensions, then the condition α may be omitted from the set (α, β, ϵ), i.e., conditions (β, ϵ) characterize an arc in a Euclidean space. For, by (β), C is an irreducible continuum between some two of its points a and b; and since, by (ϵ), C is locally connected, then by a theorem of R. L. Moore's[2] C contains an arc t from a to b. Clearly C must be identical with t. It is also true[3] that, in a Euclidean space, conditions (γ, ϵ) characterize a simple continuous arc.

G. T. WHYBURN

---

[1] Transactions of the American Mathematical Society, vol. 17 (1916), pp. 131–164.

[2] *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236.

[3] Cf. G. T. Whyburn, *Concerning connected and regular point sets*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 685–689. Although the proofs in this paper are worded for the Euclidean plane, it is obvious that they hold in a Euclidean space of any number of dimensions. Also it seems to be true that the results of this paper hold, and hence that conditions (γ, ϵ) characterize an arc, in any metric space which is locally compact.

Analytic Geometry (Revised). By CLYDE E. LOVE. The Macmillan Co., 1927. 257 pages.

In writing a text book, one should present something new, or he should present something old in a new way, or he should present something old in an old way as well or better than it has been presented before. The text under review is one which presents the usual subjects of analytic geometry in the usual way. However it should be added immediately that this usual way has been given new life and vitality by means of the arrangement of topics and well selected exercises, and by means of proper emphasis on the more important ideas. It is a text that will greatly please the majority of mathematics teachers and ought to enjoy a wide popularity. Approximately one third of the text is given to the discussion of coordinates, the straight line and the circle; one third to conic sections; and one third to solid geometry. The amount of space given to conics seems too large but the subject is well treated. The excellent exercises throughout the book are so chosen that the student will be encouraged in analytic methods and self-reliance. It is an excellent text.

R. P. STEPHENS

Analytic Geometry. By T. E. MASON and C. T. HAZARD. Boston, Ginn and Company, 1927. xi+224 pages. Price $2.40.

This is a very good elementary textbook on plane and solid analytic geometry, carefully written, published under the skillful editorial cooperation of Professor R. D. Carmichael, and well printed. It is hard to say with certainty, without having taught the book, but it seems to be a very teachable text. It is not cluttered with unimportant topics of study. The properties of conics are grouped together and not scattered. The eccentricity of conics is treated in detail. The book appears to be one from which it would be easy to select material for a course to suit most teachers. Logic and generality are not sacrificed to make the subject pleasant and easy.

The authors treat the straight line before they take up the general discussion of equation and locus, thus linking up the course with college algebra. The derivation of the general formula for the area of a triangle is given as the last problem in Chapter I, so all the problems on areas of triangles and of other polygons in this chapter are to be solved from the figures and not by a formula. No use is made of determinants in the book. The use of the projection of a line segment is carefully avoided in deriving the normal forms for the equations of straight lines in the plane and of planes in solid analytics.

The problems are numerous, excellent, and well graded as to difficulty, answers being given in the back of the book. Some of the problems would tax the best students. Many illustrative examples are worked out in the text, with most of the details given. This last is a good point, because students usually

will not fill in omitted steps in such problems. Besides many numerical problems there are plenty of a more general type, such as those about properties of geometrical figures.

No treatment is given of oblique coordinates or of the discriminants and invariants of conics and quadric surfaces. Polar coordinates and parametric coordinates are well and fully treated. There is a chapter on empirical equations. Poles and polars are discussed, also diameters. The discussion of locus problems and of curve tracing is quite systematic and not haphazard. In the treatment of these last topics, as well as in several other places in the book, the different steps of a discussion are outlined. Luckily this is not done too frequently, otherwise the student tends to follow these outlines mechanically. The ideas of function, continuity, asymptotes, plus and minus infinity could have been given with more detail and precision and not casually slipped into the text. It was a pleasure to find among the miscellaneous examples such a one as the derivation of the general formula for the centroid of a triangle. The authors show how to put the equation of a straight line in the point-slope form by purely algebraic manipulations. Why did they not do the same for the intercept form, instead of having the student substitute in a formula?

The thirty-three pages given over to solid analytics contain no general discussion of quadric surfaces, no transformation of axes, and just brief sections on cylindrical and spherical coordinates. Otherwise the treatment of lines, planes, direction cosines, and the discussion of surfaces are complete enough for an elementary book. There are not enough topics of an advanced nature either in plane or solid analytics in this book for it to be used beyond a first course.

This book might be said to hark back to the older (shall we say classic) type of analytic geometries like Fine and Thompson's *Coordinate Geometry*. This is quite a relief. For example, no calculus is used in the study of tangents and normals to conics. Also the conics and other curves are studied for their own sake. Although there are good problems from applied mathematics and although practical applications of the subject are pointed out, all these are unobtrusive. This text-book impresses one as a book on geometry and not as primarily an introduction to physics and engineering. A course chosen out of the topics there offered would lay solid foundations for more geometrical study.

ALAN D. CAMPBELL

The Mathematical Theory of Elasticity. Fourth Edition. By A. E. H. LOVE. Cambridge, England, University Press, 1927. xviii+643 pages. Price £2.

The fact that Professor Love's book has been for years the standard treatise on elasticity is in itself enough to arouse keen interest in a fourth edition. Among those who make more than casual reference to the book, the new

edition will excite genuine enthusiasm. For as one gains acquaintance with Love's treatise, the book becomes much more than a formal text of reference; one feels its charm, and partakes somewhat of the life and spirit which the author has breathed into this remarkable work.

As we turn to the new edition, we recall the first edition of 1892, the second edition of 1906, and the third edition of 1920, and ponder on the long period of years spent in the perfecting of the book. We visualize the plan and scope of the work, and appreciate the magnitude and the difficulties of the undertaking. And we note what levels of excellence the author attains in the consummation of his task, and recognize that in this last edition Professor Love has given us a treatise of exceptional finish such as can be produced only by years of painstaking and scholarly effort.

Any appraisal of Love's *Elasticity* must inevitably take into account the history of the subject itself. Since Navier, Cauchy, and Poisson laid the foundations of the mathematical theory, just a hundred years ago, the subject has developed with ever-lengthening strides. The early growth was marked by such comprehensive treatises as those of Lamé (1852), of Clebsch (1862), and of de Saint-Venant (1883), and was summarized in the exhaustive *History* of Todhunter and Pearson (1886–1893). By the 'nineties, researches were multiplying with such rapidity that it took rare courage to contemplate a further work of comprehensive character. Yet Professor Love, catching the inspiration of his teacher Webb, dreamed of a treatise that should present in the compass of a single volume a fair picture of this extensive field in its various aspects. The subject has grown so fast in the last decade, that it now seems impossible that the future should witness another attempt at a treatise as exhaustive as Love's. All the more reason, then, that we should be profoundly grateful to Professor Love for bringing his work to perfection in a fourth edition at this time.

The rapid advance of the subject in recent years prompts mention of a feature of the present edition which may give cause for regret: although the author plans to incorporate, under certain topics, some account of, or reference to, current researches, there appear to be omissions of some importance. But Professor Love can hardly be blamed for shortcomings in this direction. Added to the increased activity, there were the difficulties of tracing the literature published just before, and during, the years 1914–1918. Then, too, a paper of genuine mathematical interest is often effectively buried, nowadays, in some out-of-the-way technical journal. Indeed, it is understandable that one should find it well-nigh impossible to keep abreast of the rapidly expanding literature.

Particularly welcome are certain timely additions, mentions, and revisions which distinguish the present edition from its predecessors. Quoting from the

preface: "The most important additions are (i) a discussion of the theory of a rectangular plate, clamped at the edges, and bent by pressure applied to one face; (ii) a discussion of the theory of the resistance of a plate to pressure, when it is so thin that the extension of the middle plane, due to deformation of that plane into a curved surface, cannot be neglected; (iii) an account of the process by which stress-strain relations are deduced from the molecular theory of a crystalline solid. . . . . The most important revision concerns the theory of the equilibrium of a sphere. It has been found possible to simplify very considerably the easier parts of that theory, and thus to lead, by comparatively elementary methods, to the most important geophysical applications of the subject." As in previous editions, Professor Love continues to advance the theory by valuable contributions of his own. Moreover, when recasting the material of another, he invariably succeeds in bringing to the topic originality of treatment and vigor of exposition.

Finally, we are well reminded by this particular treatise that both author and publisher can, if they will, concern themselves with matters of detail which are too often slighted. We have here delightful evidence of painstaking editing, careful paging, faultless typography. As an example of a book of significant content brought to perfection by the combined and unselfish efforts of author and publisher, this indeed is a work that will long reflect credit upon Professor Love and the Cambridge University Press.

CARL A. GARABEDIAN

Applied Elasticity. By J. PRESCOTT. London, Longmans, Green, and Co., 1924. vii+666 pages.

Modern engineering is becoming more and more exacting. Among other things, it demands higher unit stresses, and this in turn demands a more accurate knowledge of the properties of materials and of the laws governing the distribution of the stresses in a body. Accordingly, there is a general tendency to use greater refinement in the experimental work in the laboratory and to use a more rigorous mathematical analysis in the solution of the problems that present themselves.

Not all of the problems in engineering lend themselves to a rigorous mathematical analysis.

Most of the results obtained by the use of the elementary theory of strength of materials are approximations, and it may be of decided interest to use a more rigorous mathematical analysis and derive more exact results. Thus we may fix the limitations that should be placed upon the approximate formulas in use.

The general tendency to put engineering on a more rigorous mathematical basis is illustrated by Prescott's *Applied Elasticity*. There is very little in this

book that could not be given under the title, *The Mathematical Theory of Elasticity*. The author however "tried to see the subject from the point of view of the engineer rather from that of the mathematician." Accordingly, the mathematical theory "has been developed only so far as it seemed likely to lead to the solution of practical problems"; and "only such problems as were deemed to have a practical interest have been chosen to illustrate the Theory."

This book contains 658 pages of text subdivided into nineteen chapters and three appendices. In the first three chapters (a total of forty pages), the fundamental theorems concerning stress and strain are developed. In chapter IV, (consisting of only four pages), the *Empirical Basis of Elasticity* is discussed. In the remaining fifteen chapters, the mathematical theory is developed and is applied to the solution of such problems as have a practical interest. In a general way, these fifteen chapters may be said to be concerned with the stress and strain in rods, bars, beams, struts, thin plates, spheres and cylinders, and cylinders with thin walls; also with the vibrations of rods and rotating discs. Chapter VIII deals with *The Energy in a Strained Body*. In this chapter is developed the principle of minimum energy which later is used in obtaining approximate solutions. The application of this principle makes the solution of a number of problems comparatively simple, since it requires nothing more than simple integration. Moreover, the probable errors in the results obtained are small.

The reviewer finds the heading of the chapters a little misleading. For instance, the heading of chapter V is *The Bending of Thin Rods by Transverse Forces*. Yet, this chapter is concerned mainly with beams,—bending of beams, distribution of stress in beams, beams of uniform strength, unsymmmetrical bending of beams, the theorem of three moments, etc. A similar statement holds for some of the other chapters.

This book was written for the engineer. At least it was written from the point of view of the engineer. Now Professor Prescott is an Englishman and hence may be expected to emphasize the mathematical side of the subject. It is also a recognized fact that the mathematical training of engineering students in England and on the continent is much better than that of the engineering students in this country. It may be, therefore, that the English and continental engineers are mathematically equipped to read this book. In this country, however, comparatively few engineers have the necessary knowledge of mathematics to enable them to do so, and of these few at least some are foreign trained. This is not intended as a criticism of the book. The book is an excellent one and should be on the shelf of every one who is interested in the more rigorous mathematical analysis of problems in Engineering. It is an excellent reference book.

E. W. RETTGER

Principia Mathematica. By A. N. WHITEHEAD and B. RUSSELL. Second Edition. Vol. I, 1925, lxvi+674 pages; vol. II, 1927, xxxi+742 pages; vol. III, 1927, vi+491 pages.

In the years since the first edition was published, (vol. I, 1910; II, 1912; III, 1913), the nature of the project undertaken in Principia Mathematica has become sufficiently understood. It is the demonstration that pure mathematics is rigorously deducible from correct definitions of its fundamental concepts alone, without other assumptions except such as are sufficient for a logic which provides an adequate canon of proof in general. For the completion of this project, it was necessary not only to dispense with the usual postulates of the various branches of mathematics but also to analyze the basic concepts, usually taken as undefined ideas, in such manner that these could all be defined solely in terms of the initial concepts of logic. That is, all such notions as ordinal and cardinal number (in general), the ordinal and cardinal numbers, 1, 2, 3, . . . , ω, $\aleph$, etc., the relations $+$, $\times$, etc., the ideas involved in serial order and in the various types of series, in linear, vector, and angular measure, and so on, must all be defined in terms, eventually, of such concepts as "either . . . . or", "elementary proposition," "negative of," and "propositional function."

This seemingly impossible task was carried out for the fundamental branches, excepting geometry, in the three volumes of the first edition. The fourth volume, on geometry, has not appeared, and probably is no longer to be expected. Although part VI, in volume III, covers concepts of quantity and measurement applicable in geometrical branches, it remains a matter of conjecture how far and by what procedures geometrical postulates could have been dispensed with in this fourth volume.

In the new edition, the text of the first edition is reprinted without change, except for the correction of misprints and similar errors. Any other procedure would be most difficult, since it would disturb the whole system of backward references in proofs. Accordingly, the desirable alterations are indicated and discussed in an "Introduction to the Second Edition" and in three appendices to volume I.

These alterations fall under two heads: economies of procedure which have been proved possible by work in logic done since the first edition, and discussion of a new and less comprehensive assumption in place of the "axiom of reducibility."

The alterations of the first sort are two: substitution of the Sheffer-Nicod "stroke-function" for other undefined ideas of logic, and elimination of "asserted propositional functions."

In the first edition, the undefined ideas are eight: (1) elementary propositions (those involving no variable or unspecified term) symbolized by p, q, r, . . . , (2) propositional functions (statements involving an undetermined or

variable constituent) symbolized by ϕx, $\psi(x, y)$, etc., (3) assertion of a proposition, (4) assertion of a propositional function, (5) negation of a proposition, p, symbolized by $\sim p$, (6) disjunction, $p \vee q$, meaning exactly "At least one of the two, p and q, is true," (7) the idea "ϕx is true for all values of x," symbolized by $(x).\ \phi x$, (8) "ϕx is true for some values of x," symbolized by $(\exists x).\ \phi x$. All the logical principles necessary for mathematical proof are derived from seven symbolic postulates in terms of the above, together with certain postulates (which are really principles of logical operation) which cannot be symbolized.

Professor H. M. Sheffer showed[1] that the two undefined ideas, negation and disjunction, can be replaced by one, p/q, meaning "p and q are, one or both, false." (Sheffer, Nicod, and Russell all read this "p and q are incompatible," which is misleading, since obviously two propositions which are both false, or one true and one false, may be compatible or consistent in the ordinary sense. This relation is to "material implication," on which the logic of Principia Mathematica is based, as the ordinary meaning of inconsistency is to the "implies" of ordinary inference. If we represent "p materially implies q" by $p \supset q$ and the ordinary meaning of "p implies q" by $p < q$, then

$$(p \supset q) = (p/\sim q) \quad \text{and} \quad (p < q) = \sim (p_0 \sim q),$$

where $p_0 q$ represents the usual meaning of "p is consistent with q.")

In terms of this stroke-function, the negation of p is definable as p/p, and the disjunction of p and q as $(p/p)/(q/q)$. Following this paper of Sheffer's, Jean Nicod[2] reduced five of the symbolic postulates of *Principia Mathematica* to one,

$$[p/(q/r)]/\{[t/(t/t)]/[(s/q)/((p/s)/(p/s))]\}.$$

The rather startling result is the proof that all the propositions of mathematics (unless we must except geometry) can be derived from three symbolic postulates in terms of the idea p/q.

The second economy of procedure results from the recognition that the idea of "asserted propositional function" is unnecessary. In the first edition, a theorem such as $p \supset \sim q.\ \supset .q \supset \sim p$, "If p implies that q is false, then q implies that p is false," is regarded as a propositional function, since p and q are undetermined or variable. Now propositional functions cannot in general be asserted, being neither true nor false (*e.g.*, "x is a rational fraction" is neither true nor false until the variable x is replaced by some one of its "values"). But certain propositional functions, such as this theorem, *can* be asserted: they are true *formulas*. However, in all such cases the function is assertable only because it holds for all values of its variables. Hence any statement of the form

[1] Transactions of the American Mathematical Society, vol. 16, pp. 481–488.

[2] Proceedings of the Cambridge Philosophical Society, vol. 19, pp. 32–41.

$$p \supset \sim q . \supset . q \supset \sim p$$

is equivalent to another, of the form

$$(p) : . (q) : p \supset \sim q . \supset . q \supset \sim p,$$

"for all values of p and all values of q, if p implies that q is false, then q implies that p is false." This last statement is not a propositional function but a proposition, of the general type $(x) \cdot \phi x$. Thus all "asserted propositional functions" are in reality propositions, and any theorem of the first edition, having the form of a function, should be regarded as proposition asserting this function to hold for all values of its variables.

The recognition of this fact works another improvement, which the book does not note. Since the idea "negation of p" is symbolized by $\sim p$, we ought to be able to express the fact that a deduction-formula (let us say, $p . \supset . p \supset q$) is false, by its symbolic negation, $\sim(p . \supset . p \supset q)$. But this is not possible, because $p . \supset . p \supset q$ covers the special case $p . \supset . p \supset p$, which is true. Hence we can assert neither $p . \supset . p \supset q$ nor its symbolic contradictory; which seems incompatible with the principle that one of every pair of contradictory assertions must be true. When $p . \supset . p \supset q$ becomes $(p) : . (q) : p . \supset . p \supset q$, this difficulty disappears, because its contradictory then is

$$(\exists p) : . (\exists q) : \sim (p . \supset . p \supset q),$$

"for some values of p and some values of q, $p . \supset . p \supset q$ is false." This last is, of course, true.

The emendation in the above fashion of the logic of elementary propositions, contained in sections *1 to *5 of the first edition, is very simple. It is merely sketched in the new Introduction. The logic of propositional functions, in terms of the stroke-function, is more involved and also more fundamental. The principles of it, sufficient for all later proofs, are developed in a new section *8, to replace section *9 of the first edition. This new section is printed as Appendix A.

The second main novelty of the new edition—limitation of the use of the "axiom of reducibility"—touches one of the most recondite features of Principia Mathematica. In ordinary developments of mathematics no need is felt for the axiom of reducibility or any substitute for it, because the existence of classes in general is taken for granted if defining properties of the class can be specified. (Existence of a class does not here mean existence of members of the class: null-classes may "exist.") But the authors desired to avoid such assumption, both because it seemed theoretically doubtful and because it contributed to "vicious-circle fallacies," which are encountered in the theory of assemblages. These are such as Burali-Forti's paradox about the ordinal number of all ordinals, and the paradox of the "least indefinable (transfinite) ordinal".

Accordingly, propositions which would ordinarily be interpreted as being about some class are, in Principia Mathematica, treated as statements about some propositional function which holds for every member of the class. Thus if ϕx be "x is an ordinal number," any proposition about the class of ordinals is rendered by some statement $f(\phi x)$ about the propositional function ϕx. Moreover, the "theory of types," adopted to avoid the vicious-circle fallacies, restricts the significance of any propositional function to some one type—to individuals, or classes of individuals, or classes of such classes, and so on. The result of these two limitations—the absence of the assumption that classes exist, and the theory of types—is to render it difficult to prove anything about certain legitimate totalities such as all the classes to which a given term belongs, or all the properties of a term. For instance, "All those properties which belong to 1, and are such that if they belong to n then they belong to $n+1$, are properties of every finite number." The axiom of reducibility was assumed to obviate such difficulties. It has two cases (ed. 1, vol. I, p. 174):

$$*12\cdot 1 \vdash :(\exists f):\phi x .\equiv_{x} . f!x \qquad \text{and} \qquad *12\cdot 11 \vdash :(\exists f):\phi(x,y) .\equiv_{x,y} . f!(x,y).$$

These differ only in that $*12\cdot 1$ is for functions of one variable, $*12\cdot 11$ for functions of two. The first may be read, "For any propositional function, ϕx, there is some predicative function, $f!x$, which is always true when ϕx is true, and false when ϕx is false." A "predicative function," as nearly as can be briefly put, is a single predicate; while the predicate ϕ in the axiom may be complex. Thus what the axiom assumes is that for any compound or complex statement about a term, x, there is some single predicate which holds when the complex statement is true and fails when it is false. This allows such statements as would ordinarily be made by reference to some totality of classes to which a term belongs (or the totality of properties of the term) without allowing the notion of any totality of propositional functions which the term satisfies. This last kind of totality is what leads to the vicious-circle fallacies, and is forbidden by the theory of types.

However, this axiom of reducibility is dubious. Hence the new edition proposes for consideration the substitute assumption that "A [propositional] function can only enter into a proposition through its values" (ed. 2, Appendix C, p. 659). This substitution requires the rewriting of many proofs, particularly those concerning Frege's "ancestral relation" by which the procedure of mathematical induction is shown to be completely deductive. But all the theorems essential to mathematical induction can still be demonstrated; the revision of these proofs is carried out in Appendix B. However, a similar revision for theorems concerning the real numbers and well-ordered series in general, is not found possible. "There is, ... so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-

ordered relations. . . . . It is upon this that the theory of real numbers rests, real numbers being defined as segments of the series of rationals. . . . . If we were to regard as doubtful the proposition that the series of real numbers is Dedekindian, analysis would collapse." (Ed. 2, vol. I, p. xliv).

The real purpose both of the axiom of reducibility and of the proposed substitute is to be able to treat all the propositional functions which figure in mathematics in a manner which would be valid without further assumption if it were established that all such functions can be analyzed into purely logical relations of their ultimate constituents. The difficulty of establishing this, in Principia Mathematica, is due, I believe, in part at least, to the preconception that all logical functions are truth-functions—that is, such that their truth or falsity depends solely on the truth or falsity of their terms. (See Appendix C, *passim*) All the functions of logic are, by the method of Principia Mathematica, treated as such truth-functions. For example, the material implication, $p \supset q$, can be defined by the accompanying table, where + represents "true" and − "false." That is, "p materially implies q" fails when p is true and q false, and otherwise holds.

p	q	$p \supset q$
+	+	+
−	+	+
+	−	−
−	−	+

The conception that all mathematics is concerned only with such truth-functions, leads to difficulty, because a truth which depends, in any part, upon the *form* of propositions can *not* be determined in this fashion merely from the truth or falsity of its constituents; while if differences of form are not significant in mathematics, all equivalent propositions will be identical. If, further, logical equivalence means only equivalence of truth-value, this view leads inevitably to the conclusion that there are really only two propositions, one true and one false; and all mathematics will thus collapse into an immense tautology. Essentially this conclusion was accepted by Frege, and has been renewed by Wittgenstein: Mr. Russell[1] seems to draw back from it without being willing to abandon the premises which lead to it.

As a fact, I take it, form is of the essence of both logic and mathematics. Logical truth, such as what is ordinarily meant by "q is deducible from p," depends in part upon the form of p and q, and cannot be determined from their truth or falsity alone. If I should be correct in this, it is not a matter for surprise if a procedure which depends upon treating all mathematical assertions as logical functions of their constituents, while at the same time holding logical functions to be exclusively truth-functions, should stop short of complete success. Actually, in spite of the fact that the relations assumed as primitive are all truth-functions, the method of this work is largely dependent upon considerations of

[1] I refer to Mr. Russell because he is alone responsible for the revisions of this edition, and because I do not think Mr. Whitehead would subscribe to the content of this Appendix C.

form, and is, I believe, inconsistent with the theoretical conceptions of the nature of mathematics which are expressed in it.

In volume II, the prefatory statement of symbolic conventions is considerably expanded. In the remainder of the volume, and in volume III, I do not find any material alterations.

C. I. LEWIS

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activites should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE SIMPSON MATHEMATICS CLUB, University of Florida, Gainesville, Florida.

The Simpson Mathematics Club held seven meetings in the spring of 1927. The first meeting was taken up with the election of officers and the appointment of committees. The other meetings were as follows:

February 3. "Squaring the circle" by Mr. Brodmerkle. "Geometric properties of the parabola" by Mr. Rosser.

February 17. "History of logarithms" by Mr. Quade. "Geometric properties of the ellipse" by Mr. Rosser.

March 3. "A solution of the cubic equation" by Prof. Chandler. "Geometric properties of the hyperbola" by Mr. Rosser.

March 17 and March 31. "The transcendence of e" by Prof. Kusner.

April 14. "Mathematical logic" by Mr. Brodmerkle.

It was customary to have an open forum at the end of each meeting in which members were invited to put up interesting problems for discussion and solution.

(Report by Mr. Barkley Rosser)

THE TULANE MATHEMATICS CLUB, Tulane University, New Orleans, La.

The officers of the club for the year 1926–1927 were: Professor H. E. Buchanan, president; Mrs. H. L. Titsworth, secretary-treasurer; Professor H. L. Menuet, chairman of executive board; Mrs. W. G. Allee, Professor E. S. Kalin, members of executive board.

The program for the year 1926–1927 included the following papers:

December 2, 1926. "Problem analysis" by Miss Carrie Breehan. "The laboratory method of teaching arithmetic" by Miss Cecelia Brimna. "How we subtract" by Miss Adrienne Rayl.

February 17, 1927. "The shifting emphasis in the teaching of algebra" by Professor E. S. Calin.

May 12. "The early history of trigonometry" by Professor H. E. Buchanan.

(Report by Mrs. H. L. Titsworth)

THE ALBION MATHEMATICS CLUB, Albion College, Albion, Mich.

The following programs were given at the meetings of the club during the year 1926–1927:

October 19, 1926. "Sidelights on mathematics" by Mary Alice Parsons ('27).

November 2. "Influence of periodicals and associations on mathematics" by Dorothy Schliskey ('27).

November 16. "Transcendentals and higher degree equations" by Charles Jacobes ('27).

December 7. "Comparison of mathematics in the United States before 1800 and now" by Dorothy Kennedy ('28).

January 4, 1927. "The origin and existence of hyperspace" by Francis Moore ('28).

February 1. Election of officers. "Relation of mathematics to architecture" by Harold Wilson ('28).
February 15. "Heresy and orthodoxy in geometry" by Thelma Raidle ('29). "Fallacies" by Hugh Sebastian ('29).
March 1. "Numerical magic squares" by Raymond W. Bowers ('29).
March 15. "The development of our system of notation" by Professor Sleight.
April 5. "The Chinese calculator" by Lorraine Traut ('29). "The slide rule" by Mr. Hickox.
April 12. "Recreations in arithmetic" by Laurence Wylie ('29). "Recreations in algebra" by Luna Bachelor ('29).
May 18. Annual out-door party, and election of officers. (Report by Mr. R. W. Bowers)

The Wellesley Mathematics Club, Wellesley College, Wellesley, Mass.

The following programs were given in the year 1926–1927:
October 22, 1926. Discussion of books on the mathematics browsing shelf. "The philosophy of mathematics" by Helen Sawin. "Simplified theory, statistics, and mathematics in other lands" by Frances Hartman. "Recreation in mathematics" by Isabel MacKerracher.
November 19. Anniversary Topics: "Eratosthenes" by Blanche Weatherhead; "Edmund Gunter and Albert Girard" by Janet Durand; "Wilbur Snell" by Dorothy Beaton; "Early mathematics in American colleges" by Mary Parsons.
January 7, 1927. A study of the old mathematics books in the treasure room of the college library, conducted by Miss Lennie P. Copeland.
February 25. Problems of antiquity and their solution: "Geometric representation of the roots" by Elizabeth Mitchell; "Trisection of an Angle" by Frances Baume; "Duplication of a Cube" by Elizabeth Peek; "Spirals" by Doris Miller.
March 11. "The geometry of imaginaries" by Dr. William C. Graustein of Harvard University.
May 28. Social meeting and election of officers for the year 1927–1928: President, Leona Bayly; vice-president, Elizabeth Peek; treasurer, Mary Parsons; secretary, Doris Raine; Junior executive member, Esther Kirkbride; faculty advisor, Miss Marion Stark.

(Report by Miss Frances Hartman)

The Mathematics Club of Brown University, Providence, R. I.

The following is the program for 1927–28:
November 1, 1927. "Omar Khayyám" by Leslie Thomas Fagan, Gr. "Constructions with a parallel ruler," by Eleanor Margerum, '29. "Musical scales" by Herbert Anthony Howard, '28.
November 29. "Geometrography" by Peter Shahdan, '30. "Cypher writing" by Ethel McKechnie, '28.
January 10, 1928. "Mathematics and business" by Theodore Henry Brown, Associate Professor of Business Statistics, Harvard University.
February 21. "Porisms and dimensional units" by Albert Arnold Bennett, Professor of Mathematics, Brown University.
March 27. "Sylvester" by Louise Henderson Woodman, '29. "Units of angular measurement" by Aubrey Henderson Smith, Gr. "Curiosities in numbers" by Silvio Carosella, '29.
April 24. "Linkages" by Allan Francis Nickerson, '30. "Probabilities in the game of shooting craps" by Mary Honor Cummings, '29.
May (*Date and place to be announced later*). Picnic.
Committee on Program: Professors Archibald and Bennett; Herbert Anthony Howard, '28; Ida Allenson Noble, '28; Homer Pine Smith, '29; Louise Henderson Woodman, '29.
Committee on Arrangements: Aubrey Wilfred Landers, Jr., Gr.; Silvio Carosella, '29; Elinor Margerum, '29; Allan Francis Nickerson, '30; Julia Ayer Oldham, '28.

(Report by Professor R. C. Archibald)

The Mathematics Club of the University of Oregon, Eugene, Oregon.

The program for the year of 1926–27 was the following:
October 28, 1926. "The calculus of variations" by Assistant Professor Davis.

November 18. "The life and works of Newton" by Hubert Yearian, '27.
December 16. Social meeting.
January 20, 1927. "The smoothing of curves" by Professor Milne.
February 17. "The many uses of graphs" by Gladys Beular, '26.
April 21. "Line coordinates and point equations" by Assistant Professor Bunch. Eight new members were elected.
May 12. Mathematics club picnic.
May 26. Election of officers for the next year: President, Edna English; vice-president, Robert Jackson; secretary, Beatrice Mason; Treasurer, Eric Peterson.

(Report by Miss Mason)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3315. Proposed by Harry Langman, Arverne, L. I.

Show that

$$x^2+y^2+z^2-yz-zx-xy+u^2+v^2+w^2-vw-wu-uv \geqq 3^{1/2}\begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}$$

for all real values of the variables. Also find the conditions under which the equality obtains.

3316. Proposed by Wm. B. Campbell, Colgate University.

A right circular cylinder with a base of radius r and with an altitude $2a$, closed top and bottom, contains a fixed volume of liquid, $\pi r^2 h$. It is rotated about a horizontal axis perpendicular to the axis of the cylinder at its center. It is required to determine the maximum torque and the maximum displacement of the centroid of the liquid.

3317. Proposed by Emma M. Gibson, High School, Springfield, Mo.

The coordinates of a point are expressed in terms of two parameters, α, β, by the formulae

$$\frac{x}{a}=\frac{(\alpha+\beta)}{(\alpha-\beta)}, \quad \frac{y}{b}=\frac{(1-\alpha\beta)}{(\alpha-\beta)}, \quad \frac{z}{c}=\frac{(1+\alpha\beta)}{(\alpha-\beta)},$$

where a, b, and c are constants.

Determine the locus of the point and determine the relations of the lines $\alpha=$constant and $\beta=$constant, to the surface Also show how to express the differential equation of the lines of curvature in terms of α and β.

3318. Proposed by C. N. Mills, Normal, Ill.

For the parabola $y^2=-2p(x-k)$ find the length (in terms of p and k) of the shortest segment of a line tangent to the curve included between the coordinate axes.

UNSOLVED PROBLEMS

Readers are requested to send in solutions for the following unsolved problems which were proposed in the years 1924, 1925, 1926. The number of each problem is printed in bold face type, with the page number following.

1924

3056, 101; **3057,** 101; **3060,** 101; **3073,** 206; **3075,** 206; **3076,** 254; **3077,** 254; **3082,** 305; **3085,** 305; **3087,** 306; **3089,** 353; **3094,** 353; **3097,** 402; **3099,** 455; **3106,** 499.

1925

3111, 46; **3112,** 46; **3119,** 95; **3124,** 138; **3126,** 204; **3127,** 204; **3133,** 261; **3140,** 315; **3141,** 315; 3144, 385; **3147,** 385; **3148** (3144), 433; **3157** (3153), 520; **3158** (3154), 520.

1926

3162 (3158), 47; **3163** (3159), 47; **3167** (3163), 47; **3169** (3165), 104; **3172** (3168), 104; **3174** (3170), 104; **3180** (3176), 159; **3183** (3179), 159; **3184** (3180), 159; **3185** (3173), 228; **3186** (3174), 228; **3189** (3177), 228; **3196** (3184), 278; **3204** (3192), 338; **3209,** 385; **3229,** 525; **3231,** 525.

SOLUTIONS

3231 [1926, 525]. Proposed by R. B. Stone, Purdue University.

The root-mean-square (*R. M. S.*) of n numbers $x_1, x_2, \cdots, x_n$ is defined by the formula

$$R.M.S.=\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\right)^{1/2}.$$

(*a*) For what values of n is the *R. M. S.* of the first n integers also an integer?

(*b*) For what values of a and n is the *R. M. S.* of n successive integers

$$a,\ a+1,\ \cdots a+n-1$$

also an integer?

(*c*) Under what conditions is the *R. M. S.* of n integers also an integer?

SOLUTION BY ELIJAH SWIFT, University of Vermont.

(*a*) Using the well known formula for the sum of the squares of the first n integers and setting $R.M.S.=k$, an integer, we have

$$(n+1)(2n+1)=6k^2. \tag{1}$$

Here n must be an odd integer; and, if we set $n=6\alpha+\beta$, $\beta=1, 3, -1$, we find by trial $\beta=1$ or -1. Consider first, case (*A*), where $n=6\alpha-1$. Here $\alpha(12\alpha-1)=k^2$, and, since the two factors on the left are prime to each other, $\alpha=l^2$ and $12\alpha-1=m^2$. Hence $12l^2=m^2+1$; but this is impossible, since the square of an integer is either a multiple of 4 or such a multiple plus unity. Consider next case (*B*), $n=6\alpha+1$. Here $(3\alpha+1)(4\alpha+1)=k^2$, and the factors on the left are again prime to each other. Hence $3\alpha+1=l^2$, $4\alpha+1=m^2$, and $(2l)^2-3m^2=1$. This last equation may be solved by well known methods, and from the smallest solution all others may be derived. (See, for example, Carmichael's *Diophantine Analysis*, p. 26 ff). The three smallest solutions are

$$\begin{aligned} &l=1,\ 13,\ 181;\quad n=1,\ 337,\ 65521,\\ &m=1,\ 15,\ 209;\quad k=1,\ 13\times 15,\ 181\times 209. \end{aligned} \tag{2}$$

(*b*) Subtracting the sum of the squares of the first $a-1$ integers from that of the first $a+n-1$, dividing this difference by n and reducing the results, we find

$$2n^2 + (6a - 3)n + 6a^2 - 6a + 1 = 6k^2, \ k \text{ being an integer.} \tag{3}$$

As before we find (*A*) $n=6\alpha+1$, (*B*) $n=6\alpha-1$. For (*A*) the above equation becomes

$$3\alpha^2 + \alpha = k^2 - l^2, \quad l = a + 3\alpha. \tag{4}$$

Since $k^2-l^2\equiv 0, 1$, or $3 \pmod 4$, we must have $\alpha\equiv 0$ or $1 \pmod 4$. In either case the left side of (4) is divisible by 4, and k and l are both odd or both even. Let $k-l=2m$; then (4) becomes

$$\alpha(3\alpha + 1) = 4m(l + m). \tag{5}$$

We are now in a position to find as many solutions as we desire. For, take any admissible value for α, compute the left side of (5), take for m any factor of one-fourth of the result, and we can then find l, k and a. We note that $l>3\alpha$ from (4). Any admissible value of α as large as 5 will give at least one solution and may give several. Below is given a table of a few results.

In case (*B*) we have

$$3\alpha^2 - \alpha = k^2 - l^2, \quad l = a - 1 + 3\alpha \tag{6}$$

by similar methods; and here $\alpha\equiv 0$ or $3 \pmod 4$. Hence, if again we set $k-l=2m$, we have

$$\alpha(3\alpha - 1) = 4m(l + m). \tag{7}$$

(*A*) $n = 1, 31, 49, 55, 73, 79, 79, 97, 97, \cdots$

$a = *\ \ 4, 25, 35, 74, 90, 24, 48, 147, \cdots$

(*B*) $n = 41, 47, 65, 65, 71, 89, 89, 95, 95, \cdots$

$a = 14, 22, 10, 55, 69, 8, 120, 45, 140, \cdots$

(*c*) Regarding this case apparently nothing can be said beyond the trivial remark that the sum of the squares of the n integers must be n times the square of an integer.

Also solved by H. L. Olson, F. L. Wilmer, and the Proposer.

3257 [1927, 217]. Proposed by Albert A. Bennett, Lehigh University.

Prove that if V is any real function, of three variables, x, y, z, satisfying the condition that $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2)+(\partial^2V/\partial z^2)=0$ and expressible as a term-wise differentiable power series, and if the magnitude of the vector $(\partial V/\partial x, \partial V/\partial y, \partial V/\partial z)$ is constant, say equal to unity, then the direction of this vector is also constant. Conversely, if the direction is constant, the magnitude is constant. Note that the restriction to real functions is essential for the first part of the problem but extraneous for the second part.

This is stated without proof as apparently obvious in an old physics book.

Solution by the Proposer.

Given V, a real function expressible as a power series in x, y, z, and differentiable term by term. To prove that if

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = 1 \text{ and also } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

then V is a linear function.

Proof. Express V as a series as follows: $V=P_0+P_1+P_2+\cdots+P_n+\cdots$, where P_n is a homogeneous polynomial of degree n, in x, y, z. Transform by an orthogonal transformation so that $P_0=0$, $P_1=x$. Let partial derivatives be denoted by subscripts.

Assume for induction for some $n>2$, that $P_i\equiv 0$, for each i, $1<i<n$. This is vacuously satisfied for $n=2$. Then equating coefficients of terms of like order in the expansion of $(\partial V/\partial x)^2+(\partial V/\partial y)^2+(\partial V/\partial z)^2 =1$, we have

* Arbitrary.

$$P_{jz} \equiv 0, \quad n-1 < j < 2n-1; \qquad 2P_{2n-1\,z} + P_{ny}^2 + P_{nz}^2 \equiv 0,$$

and hence, using the second condition of the hypothesis, $P_{n\,yy}+P_{n\,zz}\equiv 0$. This in turn implies that P_n is the real part of an analytic function of $u=y+zi$, and since it is homogeneous of nth degree, the function must be $(a_0+a_1i)u^n$ where a_0, a_1 are real.

Thus $P_n + iQ_n = (a_0 + a_1i)u^n$, where $P_{ny} = Q_{nz}$, $P_{nz} = -Q_{ny}$. $\therefore\ P_{ny} - iP_{nz} = (a_0 + a_1i)nu^{n-1}$.

Also $P_{ny} + iP_{nz} = (a_0 - a_1i)n\bar{u}^{n-1}$, where $\bar{u} = y - zi$.

$$\therefore\ P_{ny}^2 + P_{nz}^2 = (a_0^2 + a_1^2)n^2(u\bar{u})^{n-1} = (a_0^2 + a_1^2)n^2(y^2 + z^2)^{n-1}.$$

Then $-2P_{2n-1}=n^2(a_0^2+a_1^2)\,(y^2+z^2)^{n-1}x+F(y,z)$. Now $P_{2n-1\,xx}+P_{2n-1\,yy}+P_{2n-1\,zz}\equiv 0$. But $P_{2n-1\,xx}\equiv 0$, and $P_{2n-1\,yy}+P_{2n-1\,zz}$ cannot vanish identically unless $a_0^2+a_1^2=0$. Since a_0 and a_1 are assumed real, $a_0=0$, and $a_1=0$. Hence $P_{2n-1\,x}\equiv 0$. This in turn implies that $P_{ny}^2+P_{nz}^2\equiv 0$, or that P_{nx}, P_{ny}, P_{nz} all vanish identically and $P_n\equiv 0$. Since the above conclusions apply also for $n=2$, the induction is complete. The transformed expression for V is merely x itself, and the original V was linear.

The converse theorem, that if $\partial V/\partial x:\partial V/\partial y:\partial V/\partial z=a:b:c$ where a, b, c are constants, and if further $\partial^2V/\partial x^2+\partial^2V/\partial y^2+\partial^2V/\partial z^2=0$, then $(\partial V/\partial x)^2+(\partial V/\partial y)^2+(\partial V/\partial z)^2$ is constant, is readily proved. Indeed, an orthogonal transformation may be used to make the continued ratio, of the form, 1:0:0. If then $\partial V/\partial y\equiv 0$, and $\partial V/\partial z\equiv 0$, V is of the form $f(x)$. Since further, $(\partial^2V/\partial x^2)+(\partial^2V/\partial y)^2+(\partial^2V/\partial z^2)=0$, it follows that this V is of the form $a+bx$, and the original function was linear in x, y, z, which serves to prove the theorem.

3258 [1927, 272]. Proposed by H. E. Arnold, Wesleyan University, Middletown, Connecticut.

The perpendiculars dropped from the vertices of a triangle upon the lines joining the feet of the angle bisectors to the orthocenter of the triangle meet the respective sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line joining the orthocenter to the center of the inscribed circle.

Solution by N. A. Court, University of Oklahoma.

The proposition may be generalized as follows:

Let P be any point in the plane of a triangle ABC, and L, M, N, the traces of the lines APL, BPM, CPN, on the sides BC, CA, AB. The perpendiculars dropped from the vertices A, B, C, respectively, upon the lines joining the orthocenter H of ABC to the points L, M, N, meet the opposite corresponding sides of the triangle in three collinear points. The line joining these three points is perpendicular to the line PH.

Proof. The orthocenter H of ABC is the center of the circle (H) for which the triangle ABC is self-polar. Hence the perpendicular from A upon HL is the polar of L with respect to (H), and the trace L' of this polar on BC is the pole of AL with respect to the same circle. Similarly for the poles M', N' of the lines BM, CN. Now since the lines AL, BM, CN, are concurrent in P, by assumption, their poles L', M', N', are collinear, and the line $L'M'N'$ is the polar of P with respect to (H), and is therefore perpendicular to PH.

In the proposed problem the point P coincides with the incenter of the triangle ABC. Any number of similar propositions may be obtained by making P coincide with other remarkable points of the triangle. For instance, if P coincides with the centroid of the triangle ABC, the line $L'M'N'$ will coincide with the orthic axis of ABC (i.e., the axis of homology of ABC and its orthic triangle). This gives a proof of 3228 [1926, 525].

From the proposition proved above we may derive several other propositions. For lack of space we shall state only one of them:

The perpendiculars dropped from the orthocenter of a triangle upon the lines joining the vertices of the triangle to the traces of a transversal on the opposite sides will meet these sides in three points such that the lines projecting them from the vertices are concurrent.

Here again we may obtain any number of special cases by making the transversal coincide with some known line. For instance, we know that the external bisectors of the angles meet the sides of the triangle in three collinear points, hence the proposition:

The lines joining the vertices of a triangle to the traces, on the opposite sides, of the perpendiculars dropped from the orthocenter upon the external bisectors of the angles are concurrent.

Also solved by PAUL WERNICKE and the PROPOSER.

NOTE BY OTTO DUNKEL, Washington University.

This brief and interesting proof may at first sight appear to suffer from the fact that there is no real circle (H) for which ABC is self-polar if the triangle has only acute angles, and that it is not desirable in a geometrical proof to employ an imaginary circle. However, by means of a slight change for this case the form of the proof remains the same. Denote by $[H]$ the circle with center H and radius $r=(AH\cdot HH_a)^{1/2}=(BH\cdot HH_b)^{1/2}=(CH\cdot HH_c)^{1/2}$, where H_a is the foot of the altitude AH. If P is any point and if p' is its polar with respect to $[H]$, we shall say that the straight line p is the polar image of P if p and p' are symmetrically placed with respect to the center H. We may also say conversely that if p is any straight line P is its pole image. If Q is any point on p then it easily follows that its polar image q passes through P. It is obvious that the line joining any point P with the center H is perpendicular to its polar image p. The polar image of each vertex of ABC is the opposite side of the triangle. Hence with this circle $[H]$ the proof follows as above replacing the terms pole and polar by pole image and polar image.

3259 [1927, 272]. Proposed by Norman Anning, University of Michigan.

Solve the cyclic set of simultaneous equations,

$$x_1 - 2x_n = x_n^2 x_1; \qquad x_{j+1} - 2x_j = x_j^2 x_{j+1},$$

where n is any positive integer greater than unity and $j=1, 2, 3, \cdots, (n-1)$.

SOLUTION BY PAUL WERNICKE, Washington, D. C.

The cyclic set of equations

$$x_{j+1} = 2x_j/(1 - x_j^2), \text{ or } x_{j+1} = 2/(1/x_j - x_j), \text{ where } x_k = x_j \text{ whenever } k \equiv j \pmod{n},$$

has for any n the obvious solutions: all x_j equal 0, i, or $-i$.

By successive substitution of $2x_j/(1-x_j^2)$ for x_{j+1} etc. until only $x_{j+n}=x_j$ appears, we obtain a resulting equation of the (2^n+1)th degree. There must be the factor $x_j(x_j^2+1)$.

A goniometric solution may be obtained by putting $x_j=\tan\phi$, whence $x_{j+1}=\tan 2\phi, \cdots, x_{j+n}=x_j=\tan 2^n\phi$; whence $\phi=k\pi/(2^n-1)$.

The 2^n-1 real values of $\tan\phi$ for $k=0, 1, \cdots, (2^n-2)$ (repeating the value already known, $x_j=0$) together with $\pm i$ constitute the complete solution of our equation of degree 2^n+1.

Also solved by the PROPOSER.

3262 [1927, 272]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through two given points, real or conjugate imaginary, to draw a circle so that the ends of one of its diameters shall lie on two given lines.

SOLUTION BY OTTO DUNKEL, Washington University.

Let the two given points be A and B. The system of circles through these two points has the line AB as its radical axis, and each circle (C) has its center C on a line CM perpendicular to AB at M, and it cuts orthogonally a given fixed circle (O) with its center O on AB. Hence the points A and B are defined by such a system of circles (C) with their centers on a given line CM and each circle of the system orthogonal to a given circle (O). If a circle of the system does not cut OM, the two points A and B are conjugate imaginaries on OM.

If the two given lines, l and m, are each parallel to OM and cut CM at D and E, the center C of the required circle is the mid-point of DE. The circle (C) is then easily constructed, except for certain positions of D and E where there is no real solution.

We now consider the case where one line l is not parallel to OM but cuts it in L. Let a circle of the system cut l in R and S. Then these two points are pairs of points of the involution on l determined

by the system of circles, and therefore $S \barwedge R$. Draw the line l' symmetric to l with respect to MC and let it cut (C) in R' and S' and OM in L'. Thus R' and S' on l' are symmetric to R and S. Let the other end of the diameter RC be P. Then PR' is perpendicular to OM, and PS is perpendicular to l. Also PR' and PS are corresponding elements of two projective pencils with their centers at infinity. For $R' \barwedge R \barwedge S$ and $R'P \barwedge R'R \barwedge SP$. Hence $R'P$ and SP cut two projective ranges of points U and V on m, and the self-corresponding points of the two ranges determine the other end of the diameter, say $\overline{U}$, on m. Then $\overline{S}$ and $\overline{R}$ are determined on l and the required diameter is $\overline{UR}$. To determine $\overline{U}$ we may construct any three points U_1, U_2, U_3 on m and their mates V_1, V_2, V_3; then the usual construction described in the Note to the solution of 3240 [1928, 46] determines $\overline{U}$. As explained in the note there may be two solutions, one solution or no solution.

The locus of P is evidently an hyperbola with asymptotes perpendicular to l and to OM. When R is at L, R' is at L', S and P are at infinity, and therefore the perpendicular to OM at L' is one asymptote. When S is at L, R and P are at infinity, and the perpendicular to l at L is the other asymptote.

In the case where l is parallel to OM the hyperbola reduces to a parabola.

Also solved by PAUL WERNICKE.

3264 [1927, 335]. Proposed by L. S. Shively, Mount Morris College.

A square is subdivided into n^2 equal squares by drawing lines paralle' .o its sides, and all the diagonals of the squares thus formed are drawn. How many triangles are the ? in the configuration?

SOLUTION BY ENRIQUE LINARES, JR., Panama.

On inspection it is found that the number of triangles which have their hypothenuses parallel to the sides of the square, and which are smallest in size, is $4n^2$. The number of those next in size is $4(n-1)n$; those next, $4(n-2)(n-1)$; then $4(n-3)(n-1)$; and so on.

The number of triangles having their hypothenuses parallel to the diagonal of the square, and smallest in size, is $4n^2$. The number of those next in size is $4(n-1)^2$; those next, $4(n-2)^2$ and so on. Hence

$$N = 4[n^2 + (n-1)n \cdots + (n-2a)(n-a) + (n-2a-1)(n-a) \cdots + n^2 + (n-1)^2 \cdots + b^2 \cdots + 1];$$

$$N = 4\left[\sum_{a=0}^{a=A}(n-2a)(n-a) + \sum_{a=0}^{a=A}(n-2a-1)(n-a) + \sum_{b=0}^{b=n} b^2\right],$$

where $A = n/2 - 1$ if n is even, and $A = (n-1)/2$ if n is odd;

$$N = 4\sum_{a=0}^{a=A}(2n^2 - n) - 4\sum_{a=0}^{a=A}(6n-1)a + 4\sum_{a=0}^{a=A}4a^2 + 4\frac{n}{6}(n+1)(2n+1).$$

If n is even, $N = 4\left[(2n^2-n)\frac{n}{2} - (6n-1)\frac{n}{4}\left(\frac{n}{2}-1\right) + \frac{4}{6}\left(\frac{n}{2}-1\right)\frac{n}{2}(n-1) + \frac{n}{6}(n+1)(2n+1)\right].$

If n is odd, $N = 4\left[(2n^2-n)\left(\frac{n+1}{2}\right) - (6n-1)\left(\frac{n+1}{4}\right)\left(\frac{n-1}{2}\right) + \frac{4}{6}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)n + \frac{n}{6}(n+1)(2n+1)\right].$

On expanding and collecting terms,

$$N = 3n^3 + \frac{9}{2}n^2 + n - \frac{1}{2} \text{ when } n \text{ is odd; and } N = 3n^3 + \frac{9}{2}n^2 + n \text{ when } n \text{ is even.}$$

These two results give the single formula,

$$N = 3n^3 + \frac{9n^2}{2} + n - \frac{1}{4}[1 + (-1)^{n+1}].$$

Also solved by MICHAEL GOLDBERG, EDWARD S. SMITH, PAUL WERNICKE, and the PROPOSER.

3267 [1927, 335]. Proposed by H. A. DoBell, Colgate University.

In a given triangle the line joining the ninepoint center and the orthocenter of the pedal triangle is parallel to the Euler line of the tangential triangle.

SOLUTION BY N. A. COURT, University of Oklahoma.

The sides of the pedal, or better, the orthic triangle (i.e., the triangle formed by the feet of the altitudes) are parallel to the sides of the tangential triangle;[1] hence the two triangles are homothetic. Now the line joining the ninepoint center of the given triangle and the orthocenter of the orthic triangle is the Euler line of the orthic triangle; therefore this line is parallel to the Euler line of the tangential triangle.

Also solved by the PROPOSER.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The ninth annual meeting of the National Council of Teachers of Mathematics was held at Boston, February 24–25, 1928. On the general program were President George B. Olds, of Amherst College, who spoke on "Mathematics and Modern Life;" Professor H. W. Tyler, of Massachusetts Institute of Technology, who spoke on "Mathematics in Modern Science;" and Professor H. E. Slaught, of the University of Chicago, who spoke on "Mathematics and Sunshine." Other speakers were: Miss Edith Clarke, of the General Electric Company, on "Mathematics in Modern Business;" Walter F. Downey, English High School, Boston; Olive A. Kee, Teachers College, Boston; Harry C. Barber, Phillips Exeter Academy; W. S. Slauch, High School of Commerce, New York; and Dr. W. D. Reeve, of Teachers College, Columbia University.

Important actions of the National Council were (1) the presentation of its third yearbook, an imposing volume of over 200 pages; (2) authorization of a committee under the chairmanship of Professor Slaught to effect the incorporation of this body under the laws of Illinois; (3) authorization for the publication of a register of members early next autumn; (4) provision in the new by-laws for branches of the National Council, to consist of clubs or associations now existing or to be formed, so as to constitute a federation of all such organizations of teachers of secondary mathematics; and (5) the definite determination to increase the membership of the Council to at least ten thousand by 1930, to which end every one of the present forty-three hundred members was urged to bring in at least ten new ones.

---

[1] See, for instance, N. A. Court, *College Geometry*, p. 85.

The new officers are: President, HARRY C. BARBER, Exeter, N. H.; Secretary, J. A. FOBERG, California, Pa.; Editorial Committee, W. D. REEVE, Columbia University, JOHN R. CLARK, New York University, and H. E. SLAUGHT, University of Chicago.

A large and enthusiastic club of teachers of secondary mathematics in New York City is engaged, under the leadership of Professor W. D. Reeve, of Columbia University, in the study of curriculum and teaching improvement. On the evening of March 3 a meeting of one-hundred-fifty members of the club was addressed by Professors David Eugene Smith and Herbert Ellsworth Slaught.

Professor H. E. Slaught of the University of Chicago has just returned from an eastern trip where he had the pleasure of seeing the final proof sheets of most of the plates for the second volume of the Ahmes Papyrus. He says: "It was interesting to see at first hand just what the difficulties have been in making these plates practically perfect. It could be done only by a highly expert Egyptologist such as Dr. Bull of the Metropolitan Museum in New York, and while it has taken far longer than we anticipated, nevertheless it seems very certain that in the end we shall not so greatly regret this delay in view of the fact that the outcome promises to be so highly satisfactory. It is even now not possible to name a definite date for the actual publication of this volume, but we have reason to believe that it may be ready before the end of the present academic year. (The first volume is already printed.) Further announcement will be made as the work progresses."

The fourth annual meeting of The History of Science Society was held at the American Museum of Natural History, New York, on November 25–26, 1927, to commemorate the bi-centenary of the death of Sir Isaac Newton. This meeting was shared in by the affiliated organizations: The American Mathematical Society, The Mathematical Association of America, The American Astronomical Society, and the American Physical Society.

The two days' program consisted of papers on various phases of Sir Issac Newton's contributions to astronomy, mathematics, physics, chemistry, religion, problems of the mint, and the development of science since his day, and on Newton's influence on the early science in the American Colonies from 1687 to 1779. Two papers were devoted to each subject and these were given by scholars of distinction—from both the United States and Canada. The program for the meeting was as follows:

"Introductory address on Newton's life and work" by Dr. DAVID EUGENE SMITH.

"Newton and optics" by Professor D. C. MILLER.

"Newton's philosophy of gravitation, with special reference to modern relativity ideas" by Professor G. D. BIRKHOFF.

"Newton's influence on the development of astro-physics" by President W. W. CAMPBELL.

"Newton and dynamics" by Professor M. I. PUPIN.

"Newton as an experimental philosopher" by Dr. P. R. HEYL.

"Developments following from Newton's work" by Professor E. W. BROWN.

"Newton's twenty years' delay in announcing the law of gravitation" by Professor FLORIAN CAJORI.

"Newton's work in alchemy and chemistry" by Doctor L. C. NEWELL.

"Newton's place in the history of religious thought" by Doctor G. S. BRETT.

"Mint problems of Newton" by Mr. G. E. ROBERTS.

"Newton's first disciple in America" by Mr. F. E. BRASCH.

There was a large exhibition of books in which portrait prints, letters, autographed documents and medallic illustrations of Sir Issac Newton and of his mathematical contemporaries, which supplemented the addresses, were arranged in suitable cases, besides twelve items of Newtonian collected works and single volumes, one of these being Newton's copy of a journal with his signature and numerous notes.

These addresses are to be published in a memorial volume under the auspices of The History of Science Society. An account of the commemoration together with abstracts of all the papers has been published in *Popular Astronomy*, and in the issue of *Science* for March 9, 1928.

The Carnegie Corporation has presented the sum of $10,000 to the American Philosophical Society, for the preparation of a series of "Source Books in the History of the Sciences." Professor D. E. SMITH, of Columbia University, is in charge of the volume of this series that is devoted to mathematics.

Professor M. I. PUPIN has been given the Washington award of the Western Society of Engineers, in recognition of his work in long distance telephony and radio communication.

The following courses in mathematics are announced for the summer of 1928:

Stanford University, June 21 to August 31. By Dr. ALFRED ERRERA (Belgium): Topics selected from the foundations of mathematics; Geometry and the elements of axiomatics. By Professor E. HILLE (Princeton): Differential equations; Theory of functions of a complex variable.

Syracuse University. In addition to the usual courses in mathematics through the calculus, the following advanced courses are offered. By Professor F. F. DECKER: The teaching of algebra and geometry in secondary schools;

Introduction to the theory of groups or Introduction to modern algebra. By Professor A. D. CAMPBELL; Projective geometry or Theory of functions of a complex variable.

Dr. ETHEL L. ANDERTON of Smith College has been appointed assistant professor of mathematics at Mount Holyoke College.

Assistant Professor GRACE BERRY has been promoted to an associate professorship of mathematics at Pomona College. This semester she is on sabbatical leave, studying at the University of Illinois.

Mr. J. ALSTON CLARK of Princeton has been appointed instructor of mathematics at Rutgers University for the coming academic year.

Dr. H. N. DAVIS, professor of mechanical engineering at Harvard University, has been elected president of the Stevens Institute of Technology.

Dr. HAZEL G. SCHOONMAKER of Cornell University has been appointed assistant professor of mathematics at New Jersey College for Women.

Assistant Professor ARTHUR K. SOLUM of St. Olaf College is continuing his graduate studies at the University of Minnesota. Mr. CLARENCE CARLSON is taking Mr. Solum's place at St. Olaf College.

Professor THOMAS E. MCKINNEY of the department of mathematics and astronomy of the University of South Dakota will retire from service at the end of the present academic year on account of blindness.

Doctor C. H. RICHARDSON, professor of mathematics at Georgetown College, Kentucky, has been appointed head of the department of mathematics at Bucknell University. Professor Richardson will take the chair formerly filled by Doctor W. C. BARTOL who retires after forty-seven years of teaching service at Bucknell.

CORRIGENDUM

The following correction should be made in the March issue of the Monthly: On page 134, in line 24, replace $f(t)dt(t-z)$ by $f(t)dt/(t-z)$.

G. E. STECHERT & CO.

(Alfred Hafner)

31-33 East 10th St., NEW YORK

DOMESTIC BOOKS—FOREIGN BOOKS

Subscriptions to Foreign Periodicals

Headquarters for Scientific Books of which a large stock is kept.

Agents for over fifty years for Colleges and Public Libraries.

RUSH ORDERS can be filled promptly by using cable or radio at small extra expense.

Our second-hand department is always in the market for sets, runs and volumes of scientific and literary periodicals and for valuable books and whole libraries.

The following are some of our own publications and reprints of rare books:

Adler, Five Place Logarithms (Sammlung Goeschen)	$ .40
Albrecht, The Logarith.-trigon. Tafeln mit fuenf Dezimalstellen	.75
Bauschinger & Peters, Logarith. Trig. Tables with 8 dec. places, 2 vols.	15.00
Boole, George, Treatise on the Calculus of Finite Differences. Reprint 1926	4.00
Bremiker, C., Tables of the Common Logarithms of Numbers and Trigonometrical Functions to Six Places of Decimals	1.50
Bruhns, C., A New Manual of Logarithms to Seven Places of Decimals. With explanatory introduction in English, French and German	2.00
Byerly, W. E., Elements of the Integral Calculus. Reprint 1926	3.00
Crelle, Calculating Tables, with English and German Text. Folio	7.50
Gauss, Five Place Complete Logarithmic & Trigonometric Tables	.75
Gow, James, A Short History of Greek Mathematics, 1923	4.00
Harkness & Morley, Introduction to Theory of Analytic Functions. Reprint 1924	5.00
Harkness & Morley, A Treatise on the Theory of Functions. Reprint 1925	5.00
Koester, Logarithms & Antilogarithms (on Cardboard)	.30
Mathews, G. B., Theory of Numbers, Part I	4.00
Muller, R., Hydroelectrical Engineering. 393 ill. 75 tables, diagrams and plate. 1921. $6.00 reduced to	3.00
Oppolzer, Th., Canon der Finsternisse, hrsg.v.d. Akademie der Wissenchaften. With 160 plates. Wien 1887. Reprint 1921. Quarto. Half-cloth	15.00
Pearson, Karl. Grammar of Science. Vol. I, Reprint 1924	4.00
Peters, Logarithmic Table to 7 Places of Decimals. German Text	10.00
Peters, New Calculating Tables for Multiplication and Division by All Numbers of from One to Four Places. Folio	7.50
Petersen, J., Methods and Theories for the Solution of Problems of Geometrical Constructions, Applied to 410 Problems. Translated by S. Haagensen	1.00
Salmon, Lessons introductory to the Modern Higher Algebra. Reprint 1925	4.00
Schroen, Seven Place Common Logarithms of the Numbers from 1 to 108,000. (This edition does not contain the Trigonometrical Tables) 202 pages	1.25
Scott, C. A. An Introductory Account of Certain Modern Ideas & Methods in Plane, Analytical Geometry. Reprint, 1925	4.00
Todhunter, J., Researches in the Calculus of Variation. Reprint 1925	4.00
Vega, G., Logarithmic Tables of Numbers and Trigonometrical Functions. W. L. F. Fischer. With Explanatory Introduction in English, French and German.	2.00
Vega, G. Ten Place Log. Table: Thesaurus Logarithmorum Completus. L. 1794. Reprint 1923. Large Quarto	12.50
Webster, A. G., Partial Differential Equations of Mathematical Physics. 1927	6.00
Webster, A. G., The Dynamics of Particles and of Rigid, Elastic and Fluid Bodies. Being Lectures on Mathematical Physics. Reprint 1922	5.00
Whitworth, W. A., Choice and Chance. Reprint of fourth edition, 1925	2.50
Whitworth, W. A., Choice and Chance. Supplement to 4th edition	1.50
Whitworth, W. A., Choice and Chance. Reprint of 5th edition	4.00

CONTENTS

Uses of Punched Card Equipment in Mathematics. By GEORGE W. SNEDECOR 161

The Law of Small Numbers. By A. R. CRATHORNE 169

A New Formula for Volume. By J. B. REYNOLDS 175

Note on the Geometric Description of Linear Families of Conics by Means of Apolarity. By ALAN D. CAMPBELL 178

QUESTIONS AND DISCUSSIONS: Discussions—"An analytical discussion of problem 3221," by ROSCOE WOODS; "A method of solving a biquadratic," by C. H. SMILEY; "On the approximate division of a circumference," by TOBIAS DANTZIG; "A problem in maxima and minima," by ROGER A. JOHNSON; "On the integration of a certain type of differential equation," by JOHN P. DALTON 181

RECENT PUBLICATIONS: Reviews by G. T. WHYBURN, R. P. STEPHENS, ALAN D. CAMPBELL, CARL A. GARABEDIAN, E. W. RETTGER, C. I. LEWIS .. 191

UNDERGRADUATE MATHEMATICS CLUBS: Club activities 205

PROBLEMS AND SOLUTIONS: Problems for solution—**3315–3318.** A list of unsolved problems. Solutions—**3231, 3257, 3258, 3259, 3262, 3264, 3267** 207

NOTES AND NEWS 213

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 3.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

SOUTHERN CALIFORNIA, Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

161
169
175
178
181
191
205
207
213

V. H.
York,
URER
1928.
r.
cem-
il 20-
h.
unior
1. 28.
TICS.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty-eight persons and two institutions have been elected to membership, on applications duly certified. These are largely the result of the work of the Committee on Membership.

To Individual Membership

R. LUCILE ANDERSON, A.B. (Mt. Holyoke). Instr., Univ. of Rochester, Rochester, N. Y.

SISTER M. BERTRAND (WALTON), Ph.D. (Fordham). Teacher, Marywood Coll., Scranton, Pa.

DOROTHY F. BRIGGS, A.M. (Illinois). Teacher. Northeast Jr. High School, Kansas City, Mo.

RUTH DEWEY, M.S. (Iowa State). Instr., Western Coll., Oxford, Ohio.

L. L. DINES, Ph.D. (Chicago). Prof., Univ. of Saskatchewan, Saskatoon, Canada.

H. L. DORWART, A.B. (Washington and Jefferson). Instr., Yale Univ., New Haven, Conn.

G. B. DRUMMOND, Grad. U. S. Milit. Acad. Grad. Student, La. State Univ., Baton Rouge, La.

W. E. GENTZLER, A.M. (Columbia Univ.). Asst., Columbia Univ., New York, N. Y.

S. O. GRIMM, A.M., (Lebanon Valley), Physics, Lebanon Valley Coll., Annville, Pa.

SOLOMON GUTTMAN, 1801 Sixth Ave. N., Minneapolis, Minn.

LOUISE L. HANDY, A.B. (Michigan). Fellow in Math., Univ. of Southern Calif. 1764 Freeman Ave., Long Beach, Calif.

J. W. HURST, Ph.D. (Illinois). Asst. Prof., Montana State Coll., Bozeman, Mont.

H. S. KALTENBORN, Senior, Carnegie Inst. of Tech., Pittsburgh, Pa.

K. D. KELLY, M.S. (Chicago). Instr., Adelbert Coll., Cleveland, Ohio.

S. A. LEPESHKIN, A.M. (Columbia). Asst., Brown University, Providence, R. I.

R. E. LOWNEY, A. B. (Intermountain Union Coll.). Instr., Michigan State Coll., E. Lansing, Mich.

I. D. MACINTOSH, B.S. (Rhode Island State). Instr., Lake Forest Coll., Lake Forest, Ill.

R. R. MIDDLEMISS, B.S. in Ch.E. (Colorado). Instr., Eng. Mech., Univ. of Colorado, Boulder, Col.

C. O. MOORE, A.B. (Okla. City Univ.). Bethany-Peniel Coll., Bethany, Okla.

T. S. PETERSON, B.S. (Calif. Inst. of Tech.). Asst., Ohio State Univ., Columbus, Ohio.

J. J. QUINN, Ph.D. (Villanova). Dean, Coll. of Eng., St. Edwards Univ., Austin, Texas.

L. L. RABE, A.M. (Illinois). Instr., State Teachers' Coll., Pittsburg, Kans.

E. D. RAINVILLE, Student, Univ. of Colorado, Boulder, Colo.

H. W. RAUDENBUSH, A.M. (Columbia). Instr., Columbia Univ., New York, N. Y.

A. W. RECHT, A.M. (Denver). Instr., Math. and Astr., Univ. of Denver, Denver, Colo.

W. F. REYNOLDS, A.B. (Johns Hopkins). Mathematician, U. S. Coast & Geol. Survey, Washington, D.C.

C. E. SCHROEDER, A.M. (Boston Coll.). Asso. Prof., Boston Coll., Boston, Mass.

C. H. SMILEY, Ph.D. (California). Instr., Univ. of Illinois, Urbana, Ill.

O. T. SNODGRASS, M.S. (Iowa). Instr., Purdue Univ., W. LaFayette, Ind.

ELVA E. STARR, A.M. (Illinois). Asst. Prof., Alfred Univ., Alfred, N. Y.

J. H. STEHN, M.S. (Iowa). Instr., Univ. of Okla., Norman, Okla.

J. F. THOMSON, M.S. (Union Coll.). Instr., Tulane Univ., New Orleans, La.

J. E. TREVOR, Ph.D. (Leipzig). Prof., Thermodynamics, Cornell Univ., Ithaca, N.Y.

E. H. WAGNER, A.M. (Illinois). Instr., S. Dak. State Coll., Brookings, S. Dak.

C. M. WALSH, A.B. (Harvard). Bellport, L.I., N. Y.

G. W. WISHARD, B.S. (Nat'l Normal Univ.). 5134 Carthage Ave., Norwood, Ohio.

FERNA WRESTLER, A.M. (Kansas). Teacher, Jr. Coll., El Dorado, Kas.

MARGARET M. YOUNG, A.M. (Columbia). Instr., Hunter Coll., New York, N.Y.

To Institutional Membership

UNIVERSITY OF WICHITA, Wichita, Kansas.

PROVIDENCE COLLEGE, Providence, R. I.

There have been nominated to the American Mathematical Society the following for membership for the year under the Association sustaining membership:

H. W. BAILEY, Ph.D. (Illinois) Instr., Univ. of Illinois, Urbana, Ill.

Mrs. MARY HEGELER CARUS, La Salle, Ill.

C. C. CRAIG, Ph.D. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.

W. C. RISSELMAN, A.M. (Minnesota). Teaching Asst., Univ. of Minn., Minneapolis, Minn.

C. C. WYLIE, Ph.D. (Illinois). Asst. Prof., Astron., Univ. of Iowa, Iowa City, Ia.

W D. CAIRNS, *Secretary-Treasurer*